

TOPOLOGICAL INVARIANTS OF ANOSOV REPRESENTATIONS

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ABSTRACT. We define new topological invariants for Anosov representations and study them in detail for maximal representations of the fundamental group of a closed oriented surface Σ into the symplectic group $\mathrm{Sp}(2n, \mathbf{R})$. In particular we show that the invariants distinguish connected components of the space of symplectic maximal representations other than Hitchin components. Since the invariants behave naturally with respect to the action of the mapping class group of Σ , we obtain from this the number of components of the quotient by the mapping class group action.

For specific symplectic maximal representations we compute the invariants explicitly. This allows us to construct nice model representations in all connected components. The construction of model representations is of particular interest for $\mathrm{Sp}(4, \mathbf{R})$, because in this case there are $-1 - \chi(\Sigma)$ connected components in which all representations are Zariski dense and no model representations were known so far. Finally, we use the model representations to draw conclusions about the holonomy of symplectic maximal representations.

1. INTRODUCTION

Let Σ be a closed oriented connected surface of negative Euler characteristic, G a connected Lie group. The obstruction to lifting a representation $\rho : \pi_1(\Sigma) \rightarrow G$ to the universal cover of G is a characteristic class of ρ which is an element of $H^2(\Sigma; \pi_1(G)) \cong \pi_1(G)$.

When G is compact it is a consequence of the famous paper of Atiyah and Bott [3] that the connected components of

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G$$

are in one-to-one correspondence with the elements of $\pi_1(G)$. When G is a complex Lie group the analogous result has been conjectured by Goldman [22, p. 559] and proved by Li [38, Theorem 0.1].

When G is a real non-compact Lie group, this correspondence between connected components of $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ and elements of $\pi_1(G)$ fails to hold. Obviously characteristic classes of representations still distinguish certain connected components of $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, but they are not sufficient to distinguish all connected components.

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Examples 1. Here are some examples of this phenomenon:

(i) For $n \geq 3$, the characteristic class of a representation of $\pi_1(\Sigma)$ into $\mathrm{SL}(n, \mathbf{R})$ is an element of $\mathbf{Z}/2\mathbf{Z}$. But the space

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}(n, \mathbf{R})) / \mathrm{PGL}(n, \mathbf{R})$$

has three connected components [31, Theorem B].

(ii) For representations of $\pi_1(\Sigma)$ into $\mathrm{PSL}(2, \mathbf{R})$ the Euler number does distinguish the $4g - 3$ connected components [22, Theorem A]. For representations into $\mathrm{SL}(2, \mathbf{R})$ the Euler number is not sufficient to distinguish connected components, there are $2^{2g+1} + 2g - 3$ components, and in particular there are 2^{2g} components of maximal (or minimal) Euler number, each of which corresponds to the choice of a spin structure on Σ .

(iii) For representations of $\pi_1(\Sigma)$ into $\mathrm{Sp}(2n, \mathbf{R})$ the characteristic class which generalizes the Euler number is an element of $H^2(\Sigma; \pi_1(\mathrm{Sp}(2n, \mathbf{R}))) \cong \mathbf{Z}$. It is bounded in absolute value by $n(g - 1)$. The subspace of representations where it equals $n(g - 1)$ is called the space of maximal representations. This subspace decomposes into several connected components, 3×2^{2g} when $n \geq 3$ [18, Theorem 8.7] and $(3 \times 2^{2g} + 2g - 4)$ when $n = 2$ [24, Theorem, p. 824]. The space of maximal representations and its connected components are discussed in detail in this article.

We introduce new topological invariants for representations $\rho : \pi_1(M) \rightarrow G$, whenever ρ is an *Anosov representation*. Let us sketch the definition. Let M be a compact manifold equipped with an Anosov flow that has an invariant volume form. A representation $\rho : \pi_1(M) \rightarrow G$ is said to be a (G, H) -Anosov representation if the associated G/H -bundle over M admits a section whose image is a hyperbolic set for the induced flow on $\widetilde{M} \times_\rho G/H$ (see Section 2.1 for details). We call such a section an *Anosov section*.

Theorem 2. *Let $\rho : \pi_1(M) \rightarrow G$ be a (G, H) -Anosov representation. Then the Anosov section is unique and defines a canonical principal H -bundle over M . Hence there is a well defined map*

$$\mathrm{Hom}_{H\text{-Anosov}}(\pi_1(M), G) \longrightarrow \mathcal{B}_H(M),$$

where $\mathrm{Hom}_{H\text{-Anosov}}(\pi_1(M), G)$ denotes the subspace of (G, H) -Anosov representations and $\mathcal{B}_H(M)$ the set of gauge isomorphism classes of principal H -bundles over M . This map is continuous and natural with respect to:

- taking covers of M ,
- certain morphisms of pairs $(G, H) \rightarrow (G', H')$ (see Lemmas 2.10 and 2.11).

As a consequence the topological type of the H -bundle canonically given by the Anosov section gives rise to topological invariants of ρ . Some general properties of the invariants associated with Anosov sections are discussed in Section 4.2.

1.1. Maximal representations into $\mathrm{Sp}(2n, \mathbf{R})$. Our main focus lies on maximal representations into $\mathrm{Sp}(2n, \mathbf{R})$. Maximal representations into $\mathrm{Sp}(2n, \mathbf{R})$ are $(\mathrm{Sp}(2n, \mathbf{R}), \mathrm{GL}(n, \mathbf{R}))$ -Anosov representations [10, Theorem 6.1]. More precisely, with respect to some hyperbolic metric on Σ , the geodesic flow is an Anosov flow on the unit tangent bundle $T^1\Sigma$ and the Liouville form is invariant. The fundamental group $\pi_1(T^1\Sigma)$ is a central extension of $\pi_1(\Sigma)$, with natural projection $\pi : \pi_1(T^1\Sigma) \rightarrow \pi_1(\Sigma)$. Let $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ be a maximal representation, then the composition $\rho \circ \pi : \pi_1(T^1\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ is a $(\mathrm{Sp}(2n, \mathbf{R}), \mathrm{GL}(n, \mathbf{R}))$ -Anosov

representation. The topological invariants obtained by Theorem 2 are the characteristic classes of a $\mathrm{GL}(n, \mathbf{R})$ -bundle over $T^1\Sigma$. We only consider the first and second Stiefel-Whitney classes $sw_1(\rho \circ \pi) \in H^1(T^1\Sigma; \mathbf{F}_2)$ and $sw_2(\rho \circ \pi) \in H^2(T^1\Sigma; \mathbf{F}_2)$.

Theorem 3. *Let $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ be a maximal representation. Then the topological invariants $sw_1(\rho) = sw_1(\rho \circ \pi) \in H^1(T^1\Sigma; \mathbf{F}_2)$ and $sw_2(\rho) = sw_2(\rho \circ \pi) \in H^2(T^1\Sigma; \mathbf{F}_2)$ are subject to the following constraints:*

(i) *The image of*

$$sw_1 : \mathrm{Hom}_{\max}(\pi_1(\Sigma), G) \longrightarrow H^1(T^1\Sigma; \mathbf{F}_2)$$

is contained in one coset of $H^1(\Sigma; \mathbf{F}_2)$.

- *For n even, $sw_1(\rho)$ is in $H^1(\Sigma; \mathbf{F}_2) \subset H^1(T^1\Sigma; \mathbf{F}_2)$,*
- *for n odd, $sw_1(\rho)$ is in $H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$.*

(ii) *The image of*

$$sw_2 : \mathrm{Hom}_{\max}(\pi_1(\Sigma), G) \longrightarrow H^2(T^1\Sigma; \mathbf{F}_2)$$

lies in the image of $H^2(\Sigma; \mathbf{F}_2) \rightarrow H^2(T^1\Sigma; \mathbf{F}_2)$.

Remark 1. The homomorphism $H^1(\Sigma; \mathbf{F}_2) \rightarrow H^1(T^1\Sigma; \mathbf{F}_2)$ is induced by the natural fibration $T^1\Sigma \rightarrow \Sigma$. The Gysin exact sequence (see Equation (B.2)) implies that it is injective and that its image is of index 2.

In the case when $n = 2$, that is, for maximal representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$, let $\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ denote the subspace of maximal representations where the first Stiefel-Whitney class vanishes. This means that the $\mathrm{GL}(2, \mathbf{R})$ -bundle over $T^1\Sigma$ admits a reduction of the structure group to $\mathrm{GL}^+(2, \mathbf{R})$, equivalently the corresponding \mathbf{R}^2 -vector bundle is orientable. A reduction of the structure group to $\mathrm{GL}^+(2, \mathbf{R})$ gives rise to an Euler class, but since an orientable bundle does not have a canonical orientation this reduction is not canonical. To circumvent this problem, we introduce an *enhanced* representation space, which involves the choice of a nontrivial element $\gamma \in \pi_1(\Sigma)$. For pairs (ρ, L_+) consisting of a maximal representation with vanishing first Stiefel-Whitney class and an *oriented* Lagrangian $L_+ \subset \mathbf{R}^4$ which is fixed by $\rho(\gamma)$, there is a well-defined Euler class (see Section 4.4).

Theorem 4. *Let $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ be a maximal representation with $sw_1(\rho) = 0$. Let $\gamma \in \pi_1(\Sigma) \setminus \{1\}$ and L_+ an oriented Lagrangian fixed by $\rho(\gamma)$. Then the Euler class $e_\gamma(\rho, L_+) \in H^2(T^1\Sigma; \mathbf{Z})$ lies in the image of $H^2(\Sigma; \mathbf{Z}) \rightarrow H^2(T^1\Sigma; \mathbf{Z})$.*

For every possible topological invariant satisfying the above constraints we construct explicit representations $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ realizing this invariant (see Section 1.3 and Section 3 for the construction of the representations and Section 5 for the calculation of the invariants). From this we deduce a lower bound on the connected components of the space $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ of maximal representations.

Proposition 5. *Let $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ be the space of maximal representations.*

- (i) *If $n \geq 3$ the space $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ has at least 3×2^{2g} connected components.*
- (ii) *The space $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ has at least $3 \times 2^{2g} + 2g - 4$ connected components.*

Our method (so far) only gives a lower bound on the number of connected components. To obtain an exact count of the number of connected components using the invariants defined here, a closer analysis for surfaces with boundary would be necessary (see [22] for the case when $n = 1$).

Fortunately, the correspondence between representations and Higgs bundles allows to use algebro-geometric methods to study the topology of $\text{Rep}(\pi_1(\Sigma), G) := \text{Hom}(\pi_1(\Sigma), G)/G$. These methods have been developed by Hitchin [30] and applied to representations into Lie groups of Hermitian type in [7, 8, 18, 19, 24] leading to the exact count mentioned above in Examples 1.(iii).

Combining Proposition 5 with this exact count we can conclude that the invariants defined here distinguish connected components other than Hitchin components.

More precisely, let $\text{Hom}_{\text{Hitchin}}(\pi_1(\Sigma), \text{Sp}(2n, \mathbf{R}))$ be the space of Hitchin representations; by definition it is the union of the connected components of $\text{Hom}(\pi_1(\Sigma), \text{Sp}(2n, \mathbf{R}))$ containing representations of the form $\phi_{\text{irr}} \circ \iota$ where $\iota : \pi_1(\Sigma) \rightarrow \text{SL}(2, \mathbf{R})$ is a discrete embedding and $\phi_{\text{irr}} : \text{SL}(2, \mathbf{R}) \rightarrow \text{Sp}(2n, \mathbf{R})$ is the irreducible representation of $\text{SL}(2, \mathbf{R})$ of dimension $2n$. Hitchin representations are maximal representations.

Theorem 6. *Let $n \geq 3$. Then the topological invariants of Theorem 3 distinguish connected components of $\text{Hom}_{\text{max}} \setminus \text{Hom}_{\text{Hitchin}}$. More precisely,*

$$\text{Hom}_{\text{max}}(\pi_1(\Sigma), \text{Sp}(2n, \mathbf{R})) \setminus \text{Hom}_{\text{Hitchin}}(\pi_1(\Sigma), \text{Sp}(2n, \mathbf{R})) \xrightarrow{sw_1, sw_2} H^1(T^1\Sigma; \mathbf{F}_2) \times H^2(T^1\Sigma; \mathbf{F}_2)$$

induces a bijection from $\pi_0(\text{Hom}_{\text{max}} \setminus \text{Hom}_{\text{Hitchin}})$ to the set of pairs satisfying the constraints of Theorem 3.

It is easy to see that, when n is even, the first Stiefel-Whitney class of a Hitchin representation vanishes, i.e. one has the inclusion $\text{Hom}_{\text{Hitchin}} \subset \text{Hom}_{\text{max}, sw_1=0}$.

Theorem 7. (i) *The Euler class defines a map*

$$\text{Hom}_{\text{max}, sw_1=0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \setminus \text{Hom}_{\text{Hitchin}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \longrightarrow H^2(T^1\Sigma; \mathbf{Z})$$

which induces a bijection from $\pi_0(\text{Hom}_{\text{max}, sw_1=0} \setminus \text{Hom}_{\text{Hitchin}})$ to the image of $H^2(\Sigma; \mathbf{Z})$ in $H^2(T^1\Sigma; \mathbf{Z})$. In particular, the Euler class distinguishes connected components in $\text{Hom}_{\text{max}, sw_1=0} \setminus \text{Hom}_{\text{Hitchin}}$.

(ii) *The components of $\text{Hom}_{\text{max}} \setminus \text{Hom}_{\text{max}, sw_1=0}$ are distinguished by the first and second Stiefel-Whitney classes, i.e. the map*

$$\text{Hom}_{\text{max}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \setminus \text{Hom}_{\text{max}, sw_1=0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \xrightarrow{sw_1, sw_2} (H^1(\Sigma; \mathbf{F}_2) \setminus \{0\}) \times H^2(\Sigma; \mathbf{F}_2)$$

induces a bijection at the level of connected components.

Remark 2. Hitchin representations are not only $(\text{Sp}(2n, \mathbf{R}), \text{GL}(n, \mathbf{R}))$ -Anosov representations, but $(\text{Sp}(2n, \mathbf{R}), A)$ -Anosov representations, where A is the subgroup of diagonal matrices [36, Theorems 4.1 and 4.2]. Applying Theorem 2 to the pair $(G, H) = (\text{Sp}(2n, \mathbf{R}), A)$ one can define first Stiefel-Whitney classes $sw_1^A(\rho)$ in $H^1(T^1\Sigma; \mathbf{F}_2)$; similarly to the above discussion these invariants are shown to belong to $H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$ and distinguish the 2^{2g} connected components of $\text{Hom}_{\text{Hitchin}}(\pi_1(\Sigma), \text{Sp}(2n, \mathbf{R}))$ (see Section 4.6.2). In Section 4.6.3 we discuss

the case of the Hitchin component of a general split real simple Lie group (see Theorem 4.20 and Theorem 4.21).

Remark 3. The existence of 2^{2g} Hitchin components is due to the center of $\mathrm{Sp}(2n, \mathbf{R})$: they all project to the same component in $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{P}\mathrm{Sp}(2n, \mathbf{R}))$. For n odd, $n \geq 3$, the abundance of non-Hitchin connected components in the space of maximal representations is also explained by the center. However, for n even, the abundance of non-Hitchin connected components pertains when we consider representations into the adjoint group. A precise statement is the following theorem (see Section 4.6.4).

Theorem 8. *If n is odd and $n \geq 3$, then the space $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{P}\mathrm{Sp}(2n, \mathbf{R}))$ has 3 connected components.*

If n is even and $n \geq 4$, then there are $2^{2g} + 2$ connected components of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{P}\mathrm{Sp}(2n, \mathbf{R}))$ that lift to $\mathrm{Sp}(2n, \mathbf{R})$.

There are $2^{2g} + 2g - 2$ connected components of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{P}\mathrm{Sp}(4, \mathbf{R}))$ that lift to $\mathrm{Sp}(4, \mathbf{R})$.

This result gives lower bounds for the number of components of the space $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{P}\mathrm{Sp}(2n, \mathbf{R}))$ when n is even.

Remark 4. According to [8, Section 7] the space $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{P}\mathrm{Sp}(4, \mathbf{R})) \cong \mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{SO}_o(2, 3))$ has $2^{2g+1} + 4g - 5$ connected components. In particular, there are $2^{2g} + 2g - 3$ components that do not lift to $\mathrm{Sp}(4, \mathbf{R})$.

Maximal representations into covers of $\mathrm{Sp}(2n, \mathbf{R})$ are also Anosov, hence we have the corresponding topological invariants. We describe their properties in Section 4.6.4. We get in particular the following

Theorem 9. *Let $\mathrm{Sp}(2n, \mathbf{R})_{(k)}$ be the connected k -fold cover of $\mathrm{Sp}(2n, \mathbf{R})$. Then the space $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})_{(k)})$ is nonempty if and only if $\chi(\Sigma)$ is a multiple of k . In that case*

$$\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})_{(k)}) \longrightarrow \mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$$

is the trivial cover of degree k^{2g} . In particular

$$\#\pi_0(\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})_{(k)})) = k^{2g} \#\pi_0(\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))).$$

1.2. The action of the mapping class group. The first and second Stiefel-Whitney classes of a maximal representation ρ do not change if ρ is conjugated by an element $\mathrm{Sp}(2n, \mathbf{R})$. Thus, they give well defined functions:

$$(1.1) \quad sw_i : \mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) \longrightarrow H^i(T^1\Sigma; \mathbf{F}_2).$$

The mapping class group $\mathrm{Mod}(\Sigma)$ acts by precomposition on Rep_{\max} ; this action is properly discontinuous [37, Theorem 1.0.2] [45, Theorem 1.1] and by Theorem 2 the map (1.1) is equivariant with respect to this action and the natural action of $\mathrm{Mod}(\Sigma)$ on $H^i(T^1\Sigma; \mathbf{F}_2)$.

For the Euler class e_γ (see Theorem 4) there is a corresponding statement of equivariance for the subgroup of $\mathrm{Mod}(\Sigma)$ fixing the homotopy class of γ .

This allows us to determine the number of connected components of the space $\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))/\mathrm{Mod}(\Sigma)$.

Theorem 10. *If $n \geq 3$, the space $\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))/\mathrm{Mod}(\Sigma)$ has 6 connected components.*

The space $\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))/\mathrm{Mod}(\Sigma)$ has $2g + 2$ connected components.

1.3. Model representations. Given two representations it is in general very difficult to determine whether they lie in the same connected component or not. The invariants defined here can be computed rather explicitly and hence allow us to decide in which connected component of $\text{Hom}_{\max}(\pi_1(\Sigma), \text{Sp}(2n, \mathbf{R}))$ a specific representation lies in. This computability enables us to give particularly nice model representations in all connected components.

An easy way to construct maximal representations $\rho : \pi_1(\Sigma) \rightarrow \text{Sp}(2n, \mathbf{R})$ is by composing a discrete embedding $\pi_1(\Sigma) \rightarrow \text{SL}(2, \mathbf{R})$ with a tight homomorphism of $\text{SL}(2, \mathbf{R})$ into $\text{Sp}(2n, \mathbf{R})$ (see [13] for the notion of tight homomorphism; here they can be characterized as the morphisms inducing multiplication by n at the level of fundamental groups). The composition with the $2n$ -dimensional irreducible representation of $\text{SL}(2, \mathbf{R})$ into $\text{Sp}(2n, \mathbf{R})$ is called an *irreducible Fuchsian representation*. Hitchin representations are precisely deformations of such representations. The composition with the diagonal embedding of $\text{SL}(2, \mathbf{R})$ into the subgroup $\text{SL}(2, \mathbf{R})^n < \text{Sp}(2n, \mathbf{R})$ is called a *diagonal Fuchsian representation*. The centralizer of the image of the diagonal embedding of $\text{SL}(2, \mathbf{R})$ is isomorphic to $\text{O}(n)$. Any representation can be twisted by a representation into its centralizer, thus any diagonal Fuchsian representation can be twisted by a representation $\pi_1(\Sigma) \rightarrow \text{O}(n)$, defining a *twisted diagonal representations*. A representation obtained by one of these constructions will be called a *standard maximal representation* (see Section 3.2).

Theorem 11. *Let $n \geq 3$. Then every maximal representation $\rho : \pi_1(\Sigma) \rightarrow \text{Sp}(2n, \mathbf{R})$ can be deformed to a standard maximal representation.*

Corollary 12. *Let $n \geq 3$. Then any maximal representation $\rho : \pi_1(\Sigma) \rightarrow \text{Sp}(2n, \mathbf{R})$ can be deformed to a maximal representation whose image is contained in a proper closed subgroup of $\text{Sp}(2n, \mathbf{R})$.*

Remark 5. This conclusion can also be obtained from [18, Section 5] because the Higgs bundles for standard maximal representations can be described quite explicitly.

Our computations of the topological invariants in Section 5 give more precise information on when a maximal representation can be deformed to an irreducible Fuchsian or a diagonal Fuchsian representation:

Corollary 13. *Let $\rho : \pi_1(\Sigma) \rightarrow \text{Sp}(2n, \mathbf{R})$ be a maximal representation. Then ρ can be deformed either to an irreducible Fuchsian representation or to a diagonal Fuchsian representation if*

- (i) for $n = 2m$, $m > 2$, $sw_1(\rho) = 0$ and $sw_2(\rho) = m \frac{\chi(\Sigma)}{2} \pmod{2}$,
- (ii) for $n = 2m + 1$, $sw_2(\rho) = m \frac{\chi(\Sigma)}{2} \pmod{2}$.

The case of $\text{Sp}(4, \mathbf{R})$ is different as Theorem 11 and Corollary 12 do not hold anymore. From the count of the connected components of $\text{Rep}_{\max}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$ in [24], one can conclude that there are $2g - 3$ exceptional components not containing any standard maximal representations.

To construct model representations in these components, we decompose $\Sigma = \Sigma_l \cup \Sigma_r$ into two subsurfaces and define a representation of $\pi_1(\Sigma)$ by amalgamation of an irreducible Fuchsian representation of $\pi_1(\Sigma_l)$ with a deformation of a diagonal Fuchsian representation of $\pi_1(\Sigma_r)$. We call these representations *hybrid representations* (see Section 3.3.1 for details).

We compute the topological invariants of these representations explicitly. Allowing the Euler characteristic of the subsurface Σ_l to vary between $3 - 2g$ and -1 , we obtain $2g - 3$ hybrid representations which exhaust the $2g - 3$ exceptional components of the space of maximal representations into $\mathrm{Sp}(4, \mathbf{R})$. We conclude

Theorem 14. *Every maximal representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ can be deformed to a standard maximal representation or a hybrid representation.*

Remark 6. To obtain Theorem 14 it is essential that we are able to compute the topological invariants explicitly. Geometrically there is no obvious reason why different hybrid representations lie in different connected components. In particular, our results on the topological invariants imply that similar constructions by amalgamation (see Section 3.3.3) give representations which can be deformed to twisted diagonal representations.

From the computations of the topological invariants we also deduce the following

Theorem 15. *Any representation in $\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ with Euler class not equal to $(g - 1)[\Sigma]$ has Zariski dense image.*

Remark 7. Here the class $[\Sigma] \in H^2(T^1\Sigma; \mathbf{Z})$ is the image of the orientation class in $H^2(\Sigma; \mathbf{Z})$ under the natural map $H^2(\Sigma; \mathbf{Z}) \rightarrow H^2(T^1\Sigma; \mathbf{Z})$; it is a torsion class of order $2g - 2$ (see Appendix B.1).

Remark 8. A similar result is proved in [9, Th. 1.1.(3)] using the theory of Higgs bundles.

1.4. Holonomies of maximal representations. A direct consequence of the fact that maximal representations $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ are $(\mathrm{Sp}(2n, \mathbf{R}), \mathrm{GL}(n, \mathbf{R}))$ -Anosov representations is that the holonomy $\rho(\gamma)$ is conjugate to an element of $\mathrm{GL}(n, \mathbf{R})$ for every $\gamma \in \pi_1(\Sigma)$. More precisely $\rho(\gamma)$ fixes two transverse Lagrangians, one, L^s , being attractive, the other being repulsive. From this it follows that the holonomy $\rho(\gamma)$ is an element of $\mathrm{GL}(L^s)$ whose eigenvalues are strictly bigger than one.

For representations in the Hitchin components we have moreover that $\rho(\gamma) \in \mathrm{GL}(L^s)$ is a regular semi-simple element [36, Prop. 3.4], [25, Prop. 8]. This does not hold for other connected components of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$. Using the description of model representations in Theorem 11 and Theorem 14 we prove

Theorem 16. *Let \mathcal{H} be a connected component of*

$$\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) \setminus \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})),$$

and let $\gamma \in \pi_1(\Sigma) \setminus \{1\}$ be an element corresponding to a simple closed curve. If $n = 2$, the genus of Σ is 2 and γ is separating, we require that \mathcal{H} is not the connected component determined by $sw_1 = 0$ and $e_\gamma = 0$. Then there exist

- (i) *a representation $\rho \in \mathcal{H}$ such that the Jordan decomposition of $\rho(\gamma)$ in $\mathrm{GL}(L^s) \cong \mathrm{GL}(n, \mathbf{R})$ has a nontrivial parabolic component.*
- (ii) *a representation $\rho' \in \mathcal{H}$ such that the Jordan decomposition of $\rho'(\gamma)$ in $\mathrm{GL}(L^s) \cong \mathrm{GL}(n, \mathbf{R})$ has a nontrivial elliptic component.*

1.5. Other maximal representations. Maximal representations $\rho : \pi_1(\Sigma) \rightarrow G$ can be defined whenever G is a Lie group of Hermitian type, and they are always (G, H) -Anosov representations [14], where H is a specific subgroup of G

(see Theorem 2.19). When G is not locally isomorphic to $\mathrm{Sp}(2n, \mathbf{R})$ there is no analogue of Hitchin representations¹ and we conjecture:

Conjecture 17. *Let G be a simple Lie group of Hermitian type. If G is not locally isomorphic to $\mathrm{Sp}(2n, \mathbf{R})$, then the topological invariants of Theorem 2 distinguish connected components of $\mathrm{Rep}_{\max}(\pi_1(\Sigma), G)$.*

If the real rank of G is n , then there is an embedding of $\mathfrak{sl}(2, \mathbf{R})^n$ into \mathfrak{g} , and it is unique up to conjugation. Thus, there is always a corresponding diagonal embedding of L , a finite cover of $\mathrm{PSL}(2, \mathbf{R})$, into G , the centralizer of which is a compact subgroup of G . In particular, one can always construct twisted diagonal representations.

Conjecture 18. *Let G be of Hermitian type. If G not locally isomorphic to $\mathrm{Sp}(2n, \mathbf{R})$, then every maximal representation $\rho : \pi_1(\Sigma) \rightarrow G$ can be deformed to a twisted diagonal representation.*

If Conjecture 18 holds the analogue of Theorem 16 will also hold.

1.6. Comparison with Higgs bundle invariants. We already mentioned that the correspondence between (reductive) representations and Higgs bundles permits to use algebro-geometric methods to study the structure of $\mathrm{Rep}(\pi_1(\Sigma), G)$, and in particular to count the number of connected components. Where these methods have been applied to study representations into Lie groups of Hermitian type, see [7, 8, 18, 19, 24], the authors associate special vector bundles to the Higgs bundles, whose characteristic classes give additional invariants for maximal representations $\pi_1(\Sigma) \rightarrow G$; then they show that for any possible value of the invariants the corresponding moduli space of Higgs bundles is non-empty and connected.

We conclude the introduction with several remarks concerning the relation between the topological invariants defined here and the invariants obtained via Higgs bundles:

(i) The Higgs bundle approach has the feature that the L^2 -norm of the Higgs field gives a Morse-Bott function on the moduli space, which allows to perform Morse theory on the representation variety. In fact, for the symplectic structure on the moduli space, this function is the Hamiltonian of a circle action, so that its critical points are exactly the fixed points of this circle action. These fixed points are Higgs bundles of a very special type, called “variations of Hodge structures”. Additional information arising from this framework allows one to read off the index of the critical submanifolds from the eigenvalues of the circle action on the tangent space at a fixed point. This is used to give an exact count of the connected components in many cases, as well as to obtain further important information about the topology of the representation variety. For more details on this strategy we refer the reader to Hitchin’s article [30] as well as to the series of papers [8, 18, 19, 24].

(ii) The invariants defined here can be computed for explicit representations; this is very difficult for the Higgs bundle invariants. The computability is essential in order to determine in which connected components specific representations lie. This is of particular interest for the $2g - 3$ exceptional connected components when $n = 2$, because in these connected components no explicit representations were known before.

¹Hitchin components can be defined for any \mathbf{R} -split semisimple Lie group (see [31, Theorem A]) and the only simple \mathbf{R} -split Lie groups of Hermitian type are the symplectic groups.

(iii) The Higgs bundle invariants depend on the choice of a complex structure X on Σ and on the choice of a square-root $K^{1/2}$ of the canonical bundle $K = T^{1,0*}X$ (i.e. by [2, Prop. 3.2] such a square root corresponds to a spin structure on Σ which is given by an element v in $H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$, see [2, p. 55]). More precisely the Higgs bundle corresponding to a representation into $\mathrm{Sp}(2n, \mathbf{R})$ is a pair (E, Φ) of a holomorphic vector bundle $E = V \oplus V^*$ (V is of rank n) and Φ a holomorphic one-form with coefficients in the endomorphisms of E of the form

$$\Phi = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

with $b : V^* \rightarrow V \otimes K$ and $c : V \rightarrow V^* \otimes K$ being symmetric [24, Eq. (2.6)]. For maximal representation c is an isomorphism [24, Prop. 3.2], hence $V \otimes K^{-1/2}$ becomes an $\mathrm{O}(n, \mathbf{C})$ -bundle over X whose Stiefel-Whitney classes $w_1(\rho, v)$ in $H^1(X, \mathbf{F}_2)$ and $w_2(\rho, v)$ in $H^2(X, \mathbf{F}_2)$ (and sometimes Euler class $e(\rho, v)$) are the Higgs bundle invariants. One can remove the dependency in the spin structure by considering V as a $\mathrm{O}(n, \mathbf{C})$ -bundle over $T\Sigma \setminus \{0\}$, the complement of the zero section. The dependency in the complex structure can also be removed, because the invariants take values in a discrete set and Teichmüller space is connected.

The Stiefel-Whitney classes defined here do not depend on any choices. In particular, they are equivariant under the action of the mapping class group of Σ and behave naturally with respect to taking finite index subgroups of $\pi_1(\Sigma)$. They are also natural with respect to tight homomorphisms.

For symplectic maximal representations there is a simple relation between the invariants defined here and those defined using Higgs bundles, although they live naturally in different cohomology groups.

Proposition 19. *Let $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ be a maximal representation. Then, for any choice of spin structure v , we have the following equality in $H^i(T^1\Sigma; \mathbf{F}_2)$:*

$$\begin{aligned} sw_1(\rho) &= w_1(\rho, v) + nv \\ sw_2(\rho) &= w_2(\rho, v) + sw_1(\rho) \smile v + (g-1) \pmod{2}. \end{aligned}$$

When $n = 2$, the suspected relation is

$$e(\rho, v) = \varepsilon e(\rho) + (g-1)$$

in $H^2(T^1\Sigma; \mathbf{Z})^{tor}$, where ε depends on the choices of orientation involved in the definition of $e(\rho)$.

The existence of such relations is not surprising since the invariants arise from the same compact Lie group ($\mathrm{O}(n, \mathbf{C})$ -bundles as well as $\mathrm{GL}(n, \mathbf{R})$ -bundles have, up to isomorphism, a unique $\mathrm{O}(n, \mathbf{R})$ -reduction, and the invariants are in fact the invariants of the underlying $\mathrm{O}(n, \mathbf{R})$ -bundle). Nevertheless, it would be interesting to provide a general proof for these relations for all maximal representations. The relations in Proposition 19 are obtained from case by case considerations for model representations.

1.7. Structure of the paper. In Section 2 we recall the definition and properties of Anosov representations and of maximal representations. Examples of such representations are discussed in Section 3. The topological invariants are defined in Section 4 and computed for symplectic maximal representations in Section 5. Section 6 discusses the action of the mapping class group; in Section 7 we derive

consequences for the holonomy of symplectic maximal representations. In Appendix A we establish several important facts about positive curves and maximal representations; in Appendix B we review some facts in cohomology, in particular for the unit tangent bundle $T^1\Sigma$.

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2. PRELIMINARIES

This section introduces the notations and definitions necessary for the rest of the paper. Section 2.1 gives the definition of Anosov representations and their basic properties, in particular the uniqueness of the *Anosov section* (Proposition 2.5). Section 2.2 recalls the needed theory of maximal representations, especially that they are Anosov (Theorem 2.19) and the gluing property (Theorem 2.20).

2.1. Anosov representations. Holonomy representations of locally homogeneous geometric structures are very special. The Thurston-Ehresmann theorem [21, p. 178] [6, Theorem 2.1] states that every deformation of such a representation can be realized through deformations of the geometric structure. The concept of *Anosov structures*, introduced by Labourie in [36, Section 2], gives a dynamical generalization of this, which is more flexible, but for which it is still possible to obtain enough rigidity of the associated representations.

2.1.1. Definition. Let

- M be a compact manifold with an Anosov flow ϕ_t ,
- G a connected semisimple Lie group and (P^s, P^u) a pair of opposite parabolic subgroups of G ,
- $H = P^s \cap P^u$ their intersection, and
- $\mathcal{F}^s = G/P^s$ (resp. $\mathcal{F}^u = G/P^u$) the flag variety associated with P^s (resp. P^u).

There is a unique open G -orbit $\mathcal{X} \subset \mathcal{F}^s \times \mathcal{F}^u$. We have $\mathcal{X} = G/H$ and as open subset of $\mathcal{F}^s \times \mathcal{F}^u$ it inherits two foliations \mathcal{E}^s and \mathcal{E}^u whose corresponding distributions are denoted by E^s and E^u , i.e. $(E^s)_{(f^s, f^u)} \cong T_{f^s}\mathcal{F}^s$ and $(E^u)_{(f^s, f^u)} \cong T_{f^u}\mathcal{F}^u$.

Definition 2.1. A flat G -bundle \mathbf{P} over (M, ϕ_t) is said to have an H -reduction σ that is *flat along flow lines* if:

- σ is a section of $\mathbf{P} \times_G \mathcal{X}$; i.e. $\sigma : M \rightarrow \mathbf{P} \times_G \mathcal{X}$ defines the H -reduction²
- the restriction of σ to every orbit of ϕ_t is locally constant with respect to the induced flat structure on $\mathbf{P} \times_G \mathcal{X}$.

The two distributions E^s and E^u on \mathcal{X} are G -invariant and hence define distributions, again denoted E^s and E^u , on $\mathbf{P} \times_G \mathcal{X}$. These two distributions are invariant by the flow, again denoted by ϕ_t , that is the lift of the flow on M by the connection.

²For details on the bijective correspondence between H -reductions and sections of $\mathbf{P} \times_G G/H$ we refer the reader to [42, Section 9.4].

To every section σ of $\mathbf{P} \times_G \mathcal{X}$ we consider the two vector bundles $\sigma^* E^s$ and $\sigma^* E^u$ on M by pulling back to M the vector bundles E^s and E^u . If furthermore σ is flat along flow lines, so that it commutes with the flow, then these two vector bundles $\sigma^* E^s$ and $\sigma^* E^u$ are equipped with a natural flow.

Definition 2.2. A flat G -bundle $\mathbf{P} \rightarrow M$ is said to be a (G, H) -Anosov bundle if:

- (i) \mathbf{P} admits an H -reduction σ that is flat along flow lines, and
- (ii) the flow ϕ_t on $\sigma^* E^s$ (resp. $\sigma^* E^u$) is contracting (resp. dilating).

We call σ an *Anosov section* or an *Anosov reduction* of \mathbf{P} .

By (ii) we mean that there exists a continuous family of norms $(\|\cdot\|_m)_{m \in M}$ on $\sigma^* E^s$ (resp. $\sigma^* E^u$) and constants $A, a > 0$ such that for any e in $(\sigma^* E^s)_m$ (resp. $(\sigma^* E^u)_m$) and for any $t > 0$ one has

$$\|\phi_t e\|_{\phi_t m} \leq A \exp(-at) \|e\|_m \quad (\text{resp. } \|\phi_{-t} e\|_{\phi_{-t} m} \leq A \exp(-at) \|e\|_m).$$

Since M is compact this definition does not depend on the norm $\|\cdot\|$ or the parametrization of ϕ_t .

Definition 2.3. A representation $\pi_1(M) \rightarrow G$ is said to be (G, H) -Anosov (or H -Anosov or simply *Anosov*) if the corresponding flat G -bundle \mathbf{P} is a (G, H) -Anosov bundle.

Remark 2.4. Note that the terminology for Anosov representations is not completely uniform. A (G, H) -Anosov representation is sometimes called a (G, \mathcal{X}) -Anosov representation or an Anosov representation with respect to the parabolic subgroup P^s or P^u .

2.1.2. Properties. From now on we assume that the Anosov flow ϕ_t has an invariant volume form. For the definition of topological invariants of Anosov representations in Section 4 the following proposition will be crucial.

Proposition 2.5. *Let $\mathbf{P} \rightarrow M$ be an Anosov bundle, then there is a unique section $\sigma : M \rightarrow \mathbf{P} \times_G \mathcal{X}$ such that properties (i) and (ii) of Definition 2.2 hold. In particular, a (G, H) -Anosov bundle admits a canonical H -reduction.*

To prove Proposition 2.5 we will use the following classical fact.

Fact 2.6. Suppose that $g \in G$ has a fixed point $f^s \in \mathcal{F}^s$ and a fixed point $f^u \in \mathcal{F}^u$ such that

- (i) f^s and f^u are in general position, i.e. (f^s, f^u) belongs to $\mathcal{X} \subset \mathcal{F}^s \times \mathcal{F}^u$; and
- (ii) the (linear) action of g on the tangent space $T_{f^s} \mathcal{F}^s$ is contracting, the action on $T_{f^u} \mathcal{F}^u$ is expanding.

Then f^s is the only attracting fixed point of g in \mathcal{F}^s . Its attracting set is the set of all flags that are in general position with f^u , and f^u is the only repelling fixed point for g in \mathcal{F}^u .

Proof of Proposition 2.5. Let σ be an Anosov section of the flat bundle $\mathbf{P} \rightarrow M$. The flow ϕ_t on M is (by hypothesis) an Anosov flow that has an invariant volume form. Because of the density of closed orbits in M [1, Theorem 3], it is sufficient to show that the restriction of σ to any closed orbit γ is uniquely determined.

We identify γ with $\mathbf{Z} \backslash \mathbf{R}$ and write the restriction of \mathbf{P} to γ as $\mathbf{P}|_\gamma = \mathbf{Z} \backslash (\mathbf{R} \times G)$ where \mathbf{Z} acts on $\mathbf{R} \times G$ by $n \cdot (t, g) = (n + t, h_\gamma^n g)$ for $h_\gamma \in G$ the holonomy of \mathbf{P} along γ .

Therefore, the restriction $\sigma|_\gamma$ is a section of $\mathbf{Z} \backslash (\mathbf{R} \times \mathcal{X})$. Since σ is locally constant along γ there exists $x = (f^s, f^u)$ in \mathcal{X} such that the lift of $\sigma|_\gamma$ to \mathbf{R} is the map $\mathbf{R} \rightarrow \mathbf{R} \times \mathcal{X}$, $t \mapsto (t, x)$. Hence f^s and f^u are h_γ -invariant. Furthermore the restrictions of $\sigma^* E^s$ (resp. $\sigma^* E^u$) to γ are in this case $\mathbf{Z} \backslash (\mathbf{R} \times T_{f^s} \mathcal{F}^s)$ (resp. $\mathbf{Z} \backslash (\mathbf{R} \times T_{f^u} \mathcal{F}^u)$) so that the contraction property of Definition 2.2.(ii) exactly translates into the assumption of Fact 2.6. This implies the uniqueness of (f^s, f^u) and hence the uniqueness of $\sigma|_\gamma$. \square

Remark 2.7. Without the assumption that the Anosov flow has an invariant volume form, it is not always true that the periodic orbits are dense in M (see [17]). So Proposition 2.5 may not hold in general.

From this uniqueness one deduces

Proposition 2.8. *Let $\pi : M' \rightarrow M$ be a finite covering of M and let $\mathbf{P} \rightarrow M$ be a flat G -bundle over M .*

Then the bundle $\pi^ \mathbf{P}$ over M' is H -Anosov if and only if \mathbf{P} is H -Anosov. In that case the Anosov H -reduction of $\pi^* \mathbf{P}$ is $\pi^* \mathbf{P}_H$ the pullback of the Anosov reduction of \mathbf{P} .*

Proof. If \mathbf{P} is Anosov, then clearly $\pi^* \mathbf{P}$ is Anosov with reduction $\pi^* \mathbf{P}_H$.

Suppose now $\pi^* \mathbf{P}$ is Anosov with reduction \mathbf{P}'_H . We must prove that \mathbf{P}'_H is the pullback of a bundle over M . It is enough to treat the case when M' is connected. Since $\pi_1(M)$ is finitely generated, the finite index subgroup $\pi_1(M')$ is contained in a finite index normal subgroup. Thus, up to taking a finite cover, we can suppose that $\pi_1(M') < \pi_1(M)$ is normal, i.e. that the finite covering $M' \rightarrow M$ is a Galois covering with group S .

For any σ in S the two bundles \mathbf{P}'_H and $\sigma^* \mathbf{P}'_H$ are Anosov reductions of $\pi^* \mathbf{P}$. Hence by the previous proposition 2.5, $\sigma^* \mathbf{P}'_H = \mathbf{P}'_H$; the bundle \mathbf{P}'_H is S -invariant. This means that \mathbf{P}'_H is the pullback $\pi^* \mathbf{P}_H$ of a bundle over M . \square

An important feature of Anosov representations is their stability under deformation.

Proposition 2.9. [36, Proposition 2.1] *The set of (G, H) -Anosov representations is open in $\text{Hom}(\pi_1(M), G)$. Moreover, the H -reduction given by the Anosov section σ depends continuously on the representation.*

2.1.3. *Constructions.* Some simple constructions allow to obtain new Anosov bundles from old ones.

Lemma 2.10. (i) *Let $\rho : \pi_1(M) \rightarrow G$ be a (G, H) -Anosov representation, where $H = P^s \cap P^u$ for two opposite parabolic subgroups in G . Let Q^s, Q^u be opposite parabolic subgroups in G such that $P^s < Q^s$ and $P^u < Q^u$. Then ρ is also a (G, H') -Anosov representation, where $H' = Q^s \cap Q^u$.*

(ii) *Let \mathbf{P} be a (G, H) -Anosov bundle over M with canonical H -reduction \mathbf{P}_H and E a flat L -bundle over M . Then the fibered product $\mathbf{P} \times E$ is a $(G \times L, H \times L)$ -Anosov bundle over M , whose canonical $(H \times L)$ -reduction is the fibered product $\mathbf{P}_H \times E$.*

(iii) Let G be a Lie group of real rank one and P an (G, H) -Anosov bundle over M with canonical H -reduction P_H ; H is the centralizer in G of a Cartan subgroup A of G . Let $f : G \rightarrow L$ be a homomorphism of Lie groups and N be the centralizer in L of $f(A)$.

Then $P \times_G L$ is an (L, N) -Anosov bundle and its canonical N -reduction is the fibered product $P_H \times_H N$.

Points (i) and (ii) of this lemma are immediate; point (iii) results from [36, Proposition 3.1]. For a more general version concerning Anosov representations under embeddings of Lie groups $G \rightarrow L$ we refer to [27]. Anosov representations are also stable under finite cover of Lie groups:

Lemma 2.11. *Let $\pi : G' \rightarrow G$ be a finite covering of Lie groups. Let $H = P^s \cap P^u$ be the intersection of two opposite parabolic subgroups of G and let $H' = \pi^{-1}(H)$.*

Then a representation $\rho : \pi_1(M) \rightarrow G'$ is (G', H') -Anosov if and only if $\pi \circ \rho$ is (G, H) -Anosov. Furthermore if P' and P are the flat principal bundles for ρ and $\pi \circ \rho$, then the two Anosov reductions are equal under the identification $P' \times_{G'} G'/H' = P'/H' \cong P/H = P \times_G G/H$.

This lemma is obvious since $G'/H' \cong G/H$.

2.1.4. *Definition in terms of the universal cover of M .* A (G, H) -Anosov bundle over M can be equivalently defined in terms of equivariant maps from the universal cover of M to $\mathcal{X} \cong G/H$. Let \widetilde{M} be the universal cover of M . Then any flat G -bundle P on M can be written as:

$$P = \pi_1(M) \backslash (\widetilde{M} \times G), \quad \gamma \cdot (\tilde{m}, g) = (\gamma \cdot \tilde{m}, \rho(\gamma)g)$$

for some representation $\rho : \pi_1(M) \rightarrow G$.

Let ϕ_t be the lift of the flow on M to \widetilde{M} . This flow lifts to $\phi_t(\tilde{m}, g) = (\phi_t(\tilde{m}), g)$, defining a flow on $\widetilde{M} \times G$.

An H -reduction σ is the same as a ρ -equivariant map

$$\tilde{\sigma} : \widetilde{M} \longrightarrow G/H \cong \mathcal{X}.$$

The section σ is flat along flow lines if and only if the map $\tilde{\sigma}$ is ϕ_t -invariant.

The contraction property of the flow is now expressed as follows:

- (i) There exists a continuous family $(\|\cdot\|_{\tilde{m}})_{\tilde{m} \in \widetilde{M}}$ such that
 - for all \tilde{m} , $\|\cdot\|_{\tilde{m}}$ is a norm on $(E^s)_{\tilde{\sigma}(\tilde{m})} \subset T_{\tilde{\sigma}(\tilde{m})}\mathcal{X}$,
 - and $(\|\cdot\|_{\tilde{m}})_{\tilde{m} \in \widetilde{M}}$ is ρ -equivariant, i.e. for all \tilde{m} in \widetilde{M} , γ in $\pi_1(M)$ and e in $(E^s)_{\tilde{\sigma}(\tilde{m})}$ one has $\|\rho(\gamma) \cdot e\|_{\gamma \cdot \tilde{m}} = \|e\|_{\tilde{m}}$.
- (ii) The flow ϕ_t is contracting, i.e. there exist $A, a > 0$ such that for any $t > 0$ and \tilde{m} in \widetilde{M} and e in $(E^s)_{\tilde{\sigma}(\tilde{m})}$ then

$$\|e\|_{\phi_t \cdot \tilde{m}} \leq A \exp(-at) \|e\|_{\tilde{m}}.$$

(Note that this makes sense because $\tilde{\sigma}(\phi_t \cdot \tilde{m}) = \tilde{\sigma}(\tilde{m})$, and thus e belongs to $(E^s)_{\tilde{\sigma}(\phi_t \cdot \tilde{m})}$).

2.1.5. *Specialization to $T^1\Sigma$.* We restrict now to the case when $M = T^1\Sigma$ is the unit tangent bundle of a closed oriented connected surface Σ of negative Euler characteristic and ϕ_t is the geodesic flow on $T^1\Sigma$ with respect to some hyperbolic metric on Σ .

Let $\partial\pi_1(\Sigma)$ be the boundary at infinity of $\pi_1(\Sigma)$. Then $\partial\pi_1(\Sigma)$ is a topological circle that comes with a natural orientation and an action of $\pi_1(\Sigma)$.

There is an equivariant identification

$$T^1\tilde{\Sigma} \cong \partial\pi_1(\Sigma)^{(3+)}$$

of the unit tangent bundle of $\tilde{\Sigma}$ with the set of positively oriented triples in $\partial\pi_1(\Sigma)$. The orbit of the geodesic flow through the point (t^s, t, t^u) is

$$\mathcal{G}_{(t^s, t, t^u)} = \mathcal{G}_{(t^s, t^u)} = \{(r^s, r, r^u) \in \partial\pi_1(\Sigma)^{(3+)} \mid r^s = t^s, r^u = t^u\},$$

and the set of geodesic leaves is parametrized by $\partial\pi_1(\Sigma)^{(2)} = \partial\pi_1(\Sigma)^2 \setminus \Delta$, the complement of the diagonal in $\partial\pi_1(\Sigma)^2$. (For more details we refer the reader to [26, Section 1.1]).

Let $\pi : T^1\Sigma \rightarrow \Sigma$ be the natural projection.

Definition 2.12. A flat G -bundle P over Σ is said to be *Anosov* if its pullback π^*P over $M = T^1\Sigma$ is Anosov.

A representation $\rho : \pi_1(\Sigma) \rightarrow G$ is *Anosov* if the composition $\pi_1(T^1\Sigma) \rightarrow \pi_1(\Sigma) \xrightarrow{\rho} G$ is an Anosov representation.

Remark 2.13. Since the (unparametrized) geodesic flow can be described topologically in terms of triples of points in $\partial\pi_1(\Sigma)$, the definition of Anosov representations *does not* depend on the choice of the hyperbolic metric on Σ .

Remark 2.14. Note that for a (G, H) -Anosov bundle P over Σ , the pullback π^*P over $T^1\Sigma$ admits a canonical H -reduction, but this H -reduction in general does not come from a reduction of P . Indeed, the invariants defined in Section 4 also give obstructions for this to happen.

Let $\rho : \pi_1(\Sigma) \rightarrow G$ be an Anosov representation and P the corresponding flat G -bundle over Σ . Using the description in Section 2.1.4 it follows that there exists a ρ -equivariant map:

$$\tilde{\sigma} : T^1\tilde{\Sigma} \longrightarrow \mathcal{X} \subset \mathcal{F}^s \times \mathcal{F}^u$$

which is invariant by the geodesic flow.

In particular we get a ρ -equivariant map

$$(\xi^s, \xi^u) : \partial\pi_1(\Sigma)^{(2)} \longrightarrow \mathcal{F}^s \times \mathcal{F}^u.$$

In view of the contraction property of $\tilde{\sigma}$ (Definition 2.2.(ii)) it is easy to see that $\xi^s(t^s, t^u)$ (resp. $\xi^u(t^s, t^u)$) depends only of t^s (resp. t^u). Hence, we obtain ρ -equivariant maps $\xi^s : \partial\pi_1(\Sigma) \rightarrow \mathcal{F}^s$, and $\xi^u : \partial\pi_1(\Sigma) \rightarrow \mathcal{F}^u$.

Corollary 2.15. *Let $\rho : \pi_1(\Sigma) \rightarrow G$ be a (G, H) -Anosov representation. Then for every $\gamma \in \pi_1(\Sigma) \setminus \{1\}$ the image $\rho(\gamma)$ is conjugate to an element in H , having a unique pair of attracting/repelling fixed points $(\xi^s(t_\gamma^s), \xi^u(t_\gamma^u)) \in \mathcal{F}^s \times \mathcal{F}^u$, where (t_γ^s, t_γ^u) denotes the pair of attracting/repelling fixed points of γ in $\partial\pi_1(\Sigma)$.*

Remark 2.16. In the situation when P^s is conjugate to P^u , there is a natural identification between \mathcal{F}^s and \mathcal{F}^u , and Proposition 2.5 implies the equality $\xi^s = \xi^u$. In this case we denote the equivariant map simply by $\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{F}$.

2.2. Maximal representations.

2.2.1. Definition and properties. Let G be an almost simple noncompact Lie group of Hermitian type, i.e. the symmetric space associated with G is an irreducible Hermitian symmetric space of noncompact type. Then $\pi_1(G)$ is a finite extension of \mathbf{Z} : $\pi_1(G)/\pi_1(G)^{tor} \cong \mathbf{Z}$, and the obstruction class of ρ (see Introduction) projects to a characteristic class $\tau(\rho) \in H^2(\Sigma; \pi_1(G)/\pi_1(G)^{tor}) \cong \mathbf{Z}$, called the *Toledo invariant* of the representation ρ . The Toledo invariant $\tau(\rho)$ is bounded in absolute value

$$|\tau(\rho)| \leq -C(G)\chi(\Sigma),$$

where $C(G)$ is an explicit constant depending only on G .

Definition 2.17. A representation $\rho : \pi_1(\Sigma) \rightarrow G$ is *maximal* if

$$\tau(\rho) = -C(G)\chi(\Sigma).$$

The space of maximal representation is denoted by $\text{Hom}_{\max}(\pi_1(\Sigma), G)$. It is a union of connected components of $\text{Hom}(\pi_1(\Sigma), G)$.

Remark 2.18. In the definition of maximal representations we choose the positive extremal value of the Toledo invariant. Yet, the Toledo number $\tau(\rho) \in \mathbf{Z}$ depends on the identification $\mathbf{Z} \cong H^2(\Sigma; \mathbf{Z})$ hence on the orientation of the surface. Thus reversing the orientation changes the Toledo number to its opposite. Therefore the space where the negative extremal value is achieved is isomorphic to the space of maximal representations.

Maximal representations have been extensively studied in recent years [7, 8, 10, 11, 12, 29, 44, 45]. They enjoy several interesting properties, e.g. maximal representations are discrete embeddings, but more importantly for our considerations is the following

Theorem 2.19. [10, 14] *A maximal representation $\rho : \pi_1(\Sigma) \rightarrow G$ is an Anosov representation. More precisely, ρ is a (G, H) -Anosov representation, where $H < G$ is the stabilizer of a pair of transverse points in the Shilov boundary of the symmetric space associated with G .*

The Toledo invariant can be defined for surface with boundary [12, Section 1.1], extending the notion of maximal representation also to this case. In our construction of maximal representations we make use of the following gluing theorem, which follows from the additivity properties of the Toledo invariant established in [12, Prop. 3.2].

Theorem 2.20. [12, Th. 1, Def. 2] *Let $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$ be the decomposition of Σ along a simple closed curve and $\pi_1(\Sigma) = \pi_1(\Sigma_1) *_{\langle \gamma \rangle} \pi_1(\Sigma_2)$ the corresponding decomposition as amalgamated product. Let $\rho_i : \pi_1(\Sigma_i) \rightarrow G$ be representations which agree on γ , and let $\rho = \rho_1 * \rho_2 : \pi_1(\Sigma) \rightarrow G$ the amalgamated representation.*

If ρ_1 and ρ_2 are maximal, then ρ is maximal.

Conversely, if ρ is maximal, then ρ_1 and ρ_2 are maximal.

2.2.2. Maximal representations into $\text{Sp}(2n, \mathbf{R})$. Let \mathbf{R}^{2n} be a symplectic vector space and $(e_i)_{1 \leq i \leq 2n}$ a symplectic basis, with respect to which the symplectic form ω is given by the anti-symmetric matrix:

$$J = \begin{pmatrix} 0 & \text{Id}_n \\ -\text{Id}_n & 0 \end{pmatrix}.$$

Let $G = \text{Sp}(2n, \mathbf{R})$.

Let $L_0^s := \text{Span}(e_i)_{1 \leq i \leq n}$ be a Lagrangian subspace and $P^s < \text{Sp}(2n, \mathbf{R})$ be the parabolic subgroup stabilizing L_0^s . The stabilizer of the Lagrangian $L_0^u = \text{Span}(e_i)_{n < i \leq 2n}$ is a parabolic subgroup P^u , it is opposite to P^s . The subgroup $H = P^s \cap P^u$ is isomorphic to $\text{GL}(n, \mathbf{R})$:

$$\begin{aligned} \text{GL}(n, \mathbf{R}) &\xrightarrow{\sim} H \subset G \\ A &\longmapsto \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix}. \end{aligned}$$

(Such isomorphisms are in one to one correspondence with symplectic bases $(\epsilon_i)_{1 \leq i \leq 2n}$ —i.e. $\omega(\epsilon_i, \epsilon_j) = \omega(e_i, e_j)$ for all i, j — for which $(\epsilon_i)_{1 \leq i \leq n}$ is a basis of L_0^s). Note that P^u is conjugate to P^s in G , so that the flag variety \mathcal{F}^s is canonically isomorphic to \mathcal{F}^u ; we will denote this homogeneous space by \mathcal{L} . The space \mathcal{L} is the Shilov boundary of the symmetric space associated with $\text{Sp}(2n, \mathbf{R})$; it can be realized as the space of Lagrangian subspaces in \mathbf{R}^{2n} and the homogeneous space $\mathcal{X} \subset \mathcal{L} \times \mathcal{L}$ is the space of pairs of transverse Lagrangians.

Let $\rho : \pi_1(\Sigma) \rightarrow \text{Sp}(2n, \mathbf{R})$ be a maximal representation and \mathbf{P} the corresponding flat principal $\text{Sp}(2n, \mathbf{R})$ -bundle over $T^1\Sigma$ and E the corresponding flat symplectic \mathbf{R}^{2n} -bundle over $T^1\Sigma$. Then ρ is an $(\text{Sp}(2n, \mathbf{R}), \text{GL}(n, \mathbf{R}))$ -Anosov representation (Theorem 2.19). The canonical $\text{GL}(n, \mathbf{R})$ -reduction of \mathbf{P} is equivalent to a continuous splitting of E into two (non-flat) flow-invariant transverse Lagrangian subbundles

$$E = L^s(\rho) \oplus L^u(\rho).$$

Notation 2.21. We call this splitting the *Lagrangian reduction* of the flat symplectic Anosov \mathbf{R}^{2n} -bundle.

Remark 2.22. The symplectic form on $L^s(\rho) \oplus L^u(\rho)$ induces a duality $L^u(\rho) \cong L^s(\rho)^*$. As a result we will only consider $L^s(\rho)$ when defining invariants for symplectic maximal representations.

The Lagrangian reduction can be directly constructed from the continuous ρ -equivariant curve $\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{L}$: for any triple $v = (t^s, t, t^u) \in \partial\pi_1(\Sigma)^{(3+)} \cong T^1\tilde{\Sigma}$, we set $(L^s(\rho))_v = \xi(t^s)$ and $(L^u(\rho))_v = \xi(t^u)$. This curve satisfies an additional positivity property which we now describe.

Let (L^s, L, L^u) be a triple of pairwise transverse Lagrangians in \mathbf{R}^{2n} ; then L can be realized as the graph of $F_L \in \text{Hom}(L^s, L^u)$.

Definition 2.23. A triple (L^s, L, L^u) of pairwise transverse Lagrangians is called *positive* if the quadratic form $\omega(\cdot, F_L(\cdot))$ on L^s is positive definite.

Definition 2.24. A curve $\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{L}$ from the boundary at infinity of $\pi_1(\Sigma)$ to the space of Lagrangians is said to be *positive*, denoted by $\xi > 0$, if for every positively oriented triple (t^s, t, t^u) in $\partial\pi_1(\Sigma)^{(3+)}$ the triple of Lagrangians $(\xi(t^s), \xi(t), \xi(t^u))$ is positive.

Important facts about the space of positive curves in \mathcal{L} are established in Appendix A.1.

Theorem 2.25. [12, Theorem 8] *Let $\rho \in \text{Hom}(\pi_1(\Sigma), \text{Sp}(2n, \mathbf{R}))$ be a maximal representation and $\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{L}$ the equivariant limit curve. Then ξ is a positive curve.*

As a consequence of Proposition 2.9 we have

Fact 2.26. The positive limit curve $\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{L}$ depends continuously on the representation.

In the following we will often switch between the three different viewpoints: $\mathrm{GL}(n, \mathbf{R})$ -reduction of P , splitting $E = L^s(\rho) \oplus L^u(\rho)$ or (positive) equivariant curve $\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{L}$.

3. EXAMPLES OF REPRESENTATIONS

In this section we provide examples of Anosov and maximal representations. In Section 3.1 we describe examples of Anosov representations. In Section 3.2 we construct maximal representations into $\mathrm{Sp}(2n, \mathbf{R})$ arising from embeddings of subgroups. In Section 3.3 we construct other examples of maximal representations based on Theorem 2.20. The maximal representations constructed in this section will be considered throughout the paper.

3.1. Anosov representations. We give examples of Anosov representations. By Proposition 2.9 every small deformation of one of these representations is again an Anosov representation.

3.1.1. Hyperbolizations. Let Σ be a connected oriented closed hyperbolic surface and $M = T^1\Sigma$ its unit tangent bundle equipped with the geodesic flow ϕ_t . Hyperbolizations give rise to discrete embeddings $\pi_1(\Sigma) \rightarrow \mathrm{PSL}(2, \mathbf{R})$ which are examples of Anosov representations (see [36, Proposition 3.1]). More generally, a discrete embedding of $\pi_1(\Sigma)$ into any finite cover L of $\mathrm{PSL}(2, \mathbf{R})$ is an Anosov representation. Since $\mathrm{PSL}(2, \mathbf{R})$ has rank one there is no choice for the parabolic subgroup.

Later we will be interested in particular in Anosov bundles arising from discrete embeddings $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$. In that case $H \cong \mathrm{GL}(1, \mathbf{R})$ is the subgroup of diagonal matrices and the H -reduction corresponds to a splitting of the flat \mathbf{R}^2 -bundle over $T^1\Sigma$ into two line bundles $L^s(\iota) \oplus L^u(\iota)$.

3.1.2. Hitchin representations. A representation of $\pi_1(\Sigma)$ into a split real semisimple Lie group G is said to be a *Hitchin representation* if it can be deformed into a representation $\pi_1(\Sigma) \xrightarrow{\iota} L \xrightarrow{\tau} G$, where L is a finite cover of $\mathrm{PSL}(2, \mathbf{R})$ and $\tau_* : \mathfrak{sl}(2, \mathbf{R}) \rightarrow \mathfrak{g}$ is the principal $\mathfrak{sl}(2, \mathbf{R})$ (see [31, Section 4] for more details), and where the homomorphism $\iota : \pi_1(\Sigma) \rightarrow L$ is a discrete embedding. For the special case when $G = \mathrm{SL}(n, \mathbf{R})$, $\mathrm{Sp}(2m, \mathbf{R})$ or $\mathrm{SO}(m, m+1)$, the embedding $\tau : \mathrm{SL}(2, \mathbf{R}) \rightarrow G$ is the n -dimensional irreducible representation of $\mathrm{SL}(2, \mathbf{R})$, where $n = 2m$ when $G = \mathrm{Sp}(2m, \mathbf{R})$ and $n = 2m+1$ when $G = \mathrm{SO}(m, m+1)$. Hitchin representations are (G, H) -Anosov, where H is the intersection of two opposite minimal parabolic subgroups [36, Theorems 4.1, 4.2], [16, Theorem 1.15]. For $G = \mathrm{SL}(n, \mathbf{R})$, $\mathrm{Sp}(2m, \mathbf{R})$ or $\mathrm{SO}(m, m+1)$, H is the subgroup of diagonal matrices, in particular, the H -reduction corresponds to a splitting of the flat \mathbf{R}^n -bundle over $T^1\Sigma$ into n line bundles.

3.1.3. Other examples.

- (i) Any quasi-Fuchsian representation $\pi_1(\Sigma) \rightarrow \mathrm{PSL}(2, \mathbf{C})$ is Anosov.
- (ii) Embed $\mathrm{SL}(2, \mathbf{R})$ into $\mathrm{PGL}(3, \mathbf{R})$ as stabilizer of a point and consider the representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R}) \rightarrow \mathrm{PGL}(3, \mathbf{R})$. Then small deformations of ρ are $(\mathrm{PGL}(3, \mathbf{R}), H)$ -Anosov where H is the subgroup of diagonal matrices. These representations were studied in [5].

(iii) Let G be a semisimple Lie group, $G' < G$ a rank one subgroup, $\Lambda < G'$ a cocompact torsionfree lattice and $N = \Lambda \backslash G'/K$, where $K < G'$ is a maximal compact subgroup. Let $M = T^1N$ be the unit tangent bundle of N . Then the composition $\pi_1(M) \rightarrow \Lambda \rightarrow G' < G$ is a (G, H) -Anosov representation, where H is the identity component of the centralizer in G of a real split Cartan subgroup in G' (compare with Lemma 2.10), see [36, Prop. 3.1].

(iv) In [4, 39] a notion of quasi-Fuchsian representations for a cocompact lattice $\Lambda < \mathrm{SO}_o(1, n)$ into $\mathrm{SO}_o(2, n)$ is introduced, and it is shown that these quasi-Fuchsian representations are Anosov representations.

3.2. Standard maximal representations. In this section we describe the construction of several maximal representations

$$\rho : \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(2n, \mathbf{R})$$

to which we will refer as *standard representations*. All these representations come from homomorphisms of $\mathrm{SL}(2, \mathbf{R})$ into $\mathrm{Sp}(2n, \mathbf{R})$, possibly twisted by a representation of $\pi_1(\Sigma)$ into the centralizer of the image of $\mathrm{SL}(2, \mathbf{R})$ in $\mathrm{Sp}(2n, \mathbf{R})$. By construction, the image of any such representation will be contained in a proper closed Lie subgroup of $\mathrm{Sp}(2n, \mathbf{R})$.³

Let us fix a discrete embedding $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$.

3.2.1. Irreducible Fuchsian representations. Consider $V_0 = \mathbf{R}_1[X, Y] \cong \mathbf{R}^2$ the space of homogeneous polynomials of degree one in the variables X and Y , endowed with the symplectic form determined by

$$\omega_0(X, Y) = 1.$$

The induced action of $\mathrm{Sp}(V_0)$ on $V = \mathrm{Sym}^{2n-1} V_0 \cong \mathbf{R}_{2n-1}[X, Y] \cong \mathbf{R}^{2n}$ preserves the symplectic form $\omega_n = \mathrm{Sym}^{2n-1} \omega_0$, explicitly

$$\omega_n(P_k, P_l) = 0 \text{ if } k + l \neq 2n - 1 \text{ and } \omega_n(P_k, P_{2n-1-k}) = \frac{(-1)^k}{(2n-1)!},$$

where $P_k = X^{2n-1-k}Y^k/k!$.

This defines the $2n$ -dimensional irreducible representation of $\mathrm{Sp}(V_0) \cong \mathrm{SL}(2, \mathbf{R})$ into $\mathrm{Sp}(V) \cong \mathrm{Sp}(2n, \mathbf{R})$,

$$\phi_{irr} : \mathrm{SL}(2, \mathbf{R}) \longrightarrow \mathrm{Sp}(2n, \mathbf{R}),$$

which, by precomposition with $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$, gives rise to an *irreducible Fuchsian representation*

$$\rho_{irr} : \pi_1(\Sigma) \longrightarrow \mathrm{SL}(2, \mathbf{R}) \longrightarrow \mathrm{Sp}(2n, \mathbf{R}).$$

Notation 3.1. For a line bundle L and a non-zero integer n , we use the notation L^n for the line bundle that is the tensor product of n copies of L when n is positive or of $-n$ copies of L^* if n is negative. By convention, L^0 is the trivial line bundle. The line bundles L^n and L^{-n} are naturally dual to each other.

³More precisely, $\rho(\pi_1(\Sigma))$ will preserve a totally geodesic tight disk in the symmetric space associated with $\mathrm{Sp}(2n, \mathbf{R})$ (the notion of tight disk is not used in this paper, the interested reader is referred to [13]).

Facts 3.2. (i) Let $L^s(\iota)$ be the line bundle over $T^1\Sigma$ associated with the embedding $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$, and $E_\iota, E_{\rho_{irr}}$ the flat symplectic bundles over $T^1\Sigma$. As $E_{\rho_{irr}} = \mathrm{Sym}^{2n-1} E_\iota$ and $E_\iota = L^s(\iota) \oplus L^u(\iota) = L^s(\iota) \oplus L^s(\iota)^* = L^s(\iota) \oplus L^s(\iota)^{-1}$, the Lagrangian reduction $L^s(\rho_{irr})$ over $T^1\Sigma$ associated with ρ_{irr} is

$$L^s(\rho_{irr}) = L^s(\iota)^{2n-1} \oplus L^s(\iota)^{2n-3} \oplus \cdots \oplus L^s(\iota).$$

(ii) When $n = 2$, let us choose the symplectic identification $(\mathbf{R}_3[X, Y], -\omega_2) \cong (\mathbf{R}^4, \omega)$ given by $X^3 = e_1, X^2Y = -e_2/\sqrt{3}, Y^3 = -e_3, XY^2 = -e_4/\sqrt{3}$ (ω was defined in Section 2.2.2). With respect to this identification the irreducible representation $\phi_{irr} : \mathrm{SL}(2, \mathbf{R}) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ is given by the following formula

$$\phi_{irr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^3 & -\sqrt{3}a^2b & -b^3 & -\sqrt{3}ab^2 \\ -\sqrt{3}a^2c & 2abc + a^2d & \sqrt{3}b^2d & 2abd + b^2c \\ -c^3 & \sqrt{3}c^2d & d^3 & \sqrt{3}cd^2 \\ -\sqrt{3}ac^2 & 2acd + bc^2 & \sqrt{3}bd^2 & 2bcd + ad^2 \end{pmatrix}.$$

In particular $\phi_{irr}(\mathrm{diag}(e^m, e^{-m})) = \mathrm{diag}(e^{3m}, e^m, e^{-3m}, e^{-m})$.

This choice has been made so that $(\phi_{irr})_* : \pi_1(\mathrm{SL}(2, \mathbf{R})) \rightarrow \pi_1(\mathrm{Sp}(4, \mathbf{R}))$ is the multiplication by 2. Note that the more immediate identification $(\mathbf{R}_3[X, Y], \omega_2) \cong (\mathbf{R}^4, \omega)$ given by $X^3 = e_1, X^2Y = -e_2/\sqrt{3}, Y^3 = e_3, XY^2 = e_4/\sqrt{3}$ would produce the morphism $\mathrm{SL}(2, \mathbf{R}) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ which is the conjugate of ϕ_{irr} by $\mathrm{diag}(1, 1, -1, -1)$ (this last element is *not* in $\mathrm{Sp}(4, \mathbf{R})$); however this morphism induces the multiplication by -2 at the level of fundamental groups.

3.2.2. *Diagonal Fuchsian representations.* Let

$$\mathbf{R}^{2n} = W_1 \oplus \cdots \oplus W_n$$

with $W_i = \mathrm{Span}(e_i, e_{n+i})$ be a symplectic splitting of \mathbf{R}^{2n} . Identifying $W_i \cong \mathbf{R}^2$, this splitting gives rise to an embedding

$$\psi : \mathrm{SL}(2, \mathbf{R})^n \longrightarrow \mathrm{Sp}(W_1) \times \cdots \times \mathrm{Sp}(W_n) \subset \mathrm{Sp}(2n, \mathbf{R}).$$

Precomposing with the diagonal embedding of $\mathrm{SL}(2, \mathbf{R}) \rightarrow \mathrm{SL}(2, \mathbf{R})^n$ we obtain the diagonal embedding

$$\phi_\Delta : \mathrm{SL}(2, \mathbf{R}) \longrightarrow \mathrm{Sp}(2n, \mathbf{R}).$$

Precomposition with $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ gives rise to a *diagonal Fuchsian representation*

$$\rho_\Delta : \pi_1(\Sigma) \longrightarrow \mathrm{SL}(2, \mathbf{R}) \longrightarrow \mathrm{Sp}(2n, \mathbf{R}).$$

Facts 3.3. (i) Let $L^s(\iota)$ be the Lagrangian line bundle over $T^1\Sigma$ associated with $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$, then the Lagrangian reduction $L^s(\rho_\Delta)$ of the flat symplectic \mathbf{R}^{2n} -bundle over $T^1\Sigma$ associated with ρ_Δ is given by

$$L^s(\rho_\Delta) = L^s(\iota) \oplus \cdots \oplus L^s(\iota).$$

(ii) When $n = 2$ and with respect to the symplectic basis $(e_i)_{i=1, \dots, 4}$ the map ψ is given by the following formula

$$\psi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & \alpha & 0 & \beta \\ c & 0 & d & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix}.$$

This choice has been made so that, for the diagonal embedding ϕ_Δ , the map $(\phi_\Delta)_*$ is the multiplication by 2.

3.2.3. Twisted diagonal representations. We now vary the construction of the previous subsection. For this note that the image $\phi_\Delta(\mathrm{SL}(2, \mathbf{R})) < \mathrm{Sp}(2n, \mathbf{R})$ has a fairly large centralizer, which is a compact subgroup of $\mathrm{Sp}(2n, \mathbf{R})$ isomorphic to $\mathrm{O}(n)$.

Remark 3.4. For any maximal representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ the centralizer of $\rho(\pi_1(\Sigma))$ is a subgroup of $\mathrm{O}(n)$. This is because the centralizer of $\rho(\pi_1(\Sigma))$ fixes pointwise the positive curve in the space of Lagrangians. In particular it will be contained in the stabilizer of one positive triple of Lagrangians which is isomorphic to $\mathrm{O}(n)$.

That the centralizer of $\phi_\Delta(\mathrm{SL}(2, \mathbf{R}))$ is precisely $\mathrm{O}(n)$ can be seen in the following way. Let (W, q) be an n -dimensional vector space equipped with a definite quadratic form q and let again $V_0 = \mathbf{R}^2$ with its standard symplectic form ω_0 . The tensor product $V_0 \otimes W$ inherits the bilinear nondegenerate form $\omega_0 \otimes q$ which is antisymmetric, so that we can choose a symplectic identification $\mathbf{R}^{2n} \cong V_0 \otimes W$. This gives an embedding

$$\mathrm{SL}(2, \mathbf{R}) \times \mathrm{O}(n) \cong \mathrm{Sp}(V_0) \times \mathrm{O}(W, q) \xrightarrow{\phi_\Delta} \mathrm{Sp}(2n, \mathbf{R}),$$

which extends the morphism ϕ_Δ defined above.

Now given $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ and a representation $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{O}(n)$, we set

$$\begin{aligned} \rho_\Theta &= \iota \otimes \Theta : \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(2n, \mathbf{R}) \\ \gamma &\longmapsto \phi_\Delta(\iota(\gamma), \Theta(\gamma)). \end{aligned}$$

We will call such a representation a *twisted diagonal representation*.

Facts 3.5. (i) The flat bundle E over Σ associated with $\rho_\Theta : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R}) \cong \mathrm{Sp}(V_0 \otimes W)$ is of the form

$$E = E_0 \otimes W,$$

where (with a slight abuse of notation) W is the flat n -plane bundle associated with Θ and E_0 the flat 2-plane bundle over Σ associated with ι .

(ii) Let $L^s(\iota)$ be the line bundle over $T^1\Sigma$ associated with ι . Let \overline{W} denote the flat n -plane bundle over $T^1\Sigma$ given by the pullback of W . Then the Lagrangian reduction $L^s(\rho_\Theta)$ is the tensor product

$$L^s(\rho_\Theta) = L^s(\iota) \otimes \overline{W}.$$

3.2.4. Standard representations for other groups. Let G be an almost simple Lie group of Hermitian type of real rank n . Then there exists a unique (up to conjugation by G) embedding $\mathfrak{sl}(2, \mathbf{R})^n \rightarrow \mathfrak{g}$, which gives, at the level of Lie groups, a finite-to-one morphism $L^n \rightarrow G$ where L is a finite cover of $\mathrm{PSL}(2, \mathbf{R})$. We call the precomposition of such an embedding with the diagonal embedding $L \rightarrow L^n$ a diagonal embedding

$$\phi_\Delta : L \longrightarrow G.$$

The centralizer of $\phi_\Delta(L)$ in G is always a compact subgroup K' of G .

The composition $\rho_\Delta = \phi_\Delta \circ \iota : \pi_1(\Sigma) \rightarrow G$ is a maximal representation. Given a representation $\Theta : \pi_1(\Sigma) \rightarrow K'$ we can again define a *twisted diagonal representation*

$$\begin{aligned} \rho_\Theta &: \pi_1(\Sigma) \longrightarrow G \\ \gamma &\longmapsto \rho_\Delta(\gamma) \cdot \Theta(\gamma). \end{aligned}$$

Remark 3.6. In the general case the subgroup K' can be characterized as being the intersection of a maximal compact subgroup K in G with the subgroup $H < G$, which is the stabilizer of a pair of transverse points in the Shilov boundary of the symmetric space associated with G . Equivalently, K' is the stabilizer in G of a maximal triple of points in the Shilov boundary. (For the definition of maximal triples see [12, Section 2.1.3]).

3.3. Amalgamated representations. Due to Theorem 2.20 we can construct maximal representations of $\pi_1(\Sigma)$ by amalgamation of maximal representations of the fundamental groups of subsurfaces.

Let $\Sigma = \Sigma_l \cup_\gamma \Sigma_r$ be a decomposition of Σ along a simple closed separating oriented geodesic γ into two subsurfaces Σ_l , lying to the left of γ , and Σ_r , lying to the right of γ . Then $\pi_1(\Sigma)$ is isomorphic to $\pi_1(\Sigma_l) *_{\langle \gamma \rangle} \pi_1(\Sigma_r)$, where we identify γ with the element it defines in $\pi_1(\Sigma)$.

We will call a representation constructed by amalgamation of two representations $\rho_l : \pi_1(\Sigma_l) \rightarrow G$ and $\rho_r : \pi_1(\Sigma_r) \rightarrow G$ with $\rho_l(\gamma) = \rho_r(\gamma)$ an *amalgamated representation* $\rho = \rho_l * \rho_r : \pi_1(\Sigma) \rightarrow G$. By Theorem 2.20, the amalgamated representation ρ is maximal if and only if ρ_l and ρ_r are maximal.⁴

3.3.1. Hybrid representations. In this section we describe the most important class of maximal representations $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ obtained via amalgamation. We call these representation *hybrid representations* to distinguish them from general amalgamated representations.

Let $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ be a discrete embedding. The basic idea of the construction of hybrid representations is to amalgamate the restriction to Σ_l of the irreducible Fuchsian representation $\rho_{irr} = \phi_{irr} \circ \iota$ and the restriction to Σ_r of the diagonal Fuchsian representation $\rho_\Delta = \phi_\Delta \circ \iota$. This does not directly work because the holonomies of ρ_{irr} and ρ_Δ along γ do not agree, but a slight modification works.

Remark 3.7. This idea of using two different embeddings of $\mathrm{SL}(2, \mathbf{R})$ to construct amalgamated representations was previously used in the paper [23] to construct representations $\pi_1(\Sigma) \rightarrow \mathrm{PU}(2, 1)$ of every possible Toledo number (see in particular the concept of “hybrid surfaces” [23, Sec. 2.4]).

Assume that $\iota(\gamma) = \mathrm{diag}(e^m, e^{-m})$ with $m > 0$. Set

$$\rho_l := \phi_{irr} \circ \iota : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R}),$$

with ϕ_{irr} defined in Facts 3.2.(ii). Then $\rho_l(\gamma) = \mathrm{diag}(e^{3m}, e^m, e^{-3m}, e^{-m})$.

Let $(\tau_{1,t})_{t \in [0,1]}$ and $(\tau_{2,t})_{t \in [0,1]}$ be continuous paths of discrete embeddings $\pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ such that $\tau_{1,0} = \tau_{2,0} = \iota$, and, for all $t \in [0, 1]$,

$$\tau_{1,t}(\gamma) = \begin{pmatrix} e^{l_{1,t}} & \\ & e^{-l_{1,t}} \end{pmatrix} \text{ and } \tau_{2,t}(\gamma) = \begin{pmatrix} e^{l_{2,t}} & \\ & e^{-l_{2,t}} \end{pmatrix},$$

where $l_{1,t} > 0$ and $l_{2,t} > 0$, $l_{1,0} = l_{2,0} = m$, $l_{1,1} = 3m$ and $l_{2,1} = m$. The existence of $\tau_{i,t}$ is a classical fact from hyperbolic geometry, for the reader's convenience we include the statement we are using in Lemma A.4. Set

$$\rho_r := \psi \circ (\tau_{1,1}, \tau_{2,1}) : \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(4, \mathbf{R}),$$

⁴Note that it is important that the Toledo invariant for both ρ_l and ρ_r are of the same sign (and also that the orientations of Σ_l and Σ_r —involved in the definition of the Toledo number— are compatible with the orientation of Σ). Amalgamating a maximal representation with a minimal representation does not give rise to a maximal representation.

with ψ defined by Facts 3.3.(ii). Then ρ_r is a continuous deformation of $\phi_\Delta \circ \iota$ which satisfies $\rho_r(\gamma) = \rho_l(\gamma)$.

Definition 3.8. A *hybrid representation* is the amalgamated representation

$$\rho := \rho_l|_{\pi_1(\Sigma_l)} * \rho_r|_{\pi_1(\Sigma_r)} : \pi_1(\Sigma) = \pi_1(\Sigma_l) *_{\langle \gamma \rangle} \pi_1(\Sigma_r) \longrightarrow \mathrm{Sp}(4, \mathbf{R}).$$

If $\chi(\Sigma_l) = k$ we call the representation a k -hybrid representation.

Since $\rho_l|_{\pi_1(\Sigma_l)}$ and $\rho_r|_{\pi_1(\Sigma_r)}$ are maximal representations, the representation ρ is maximal (see Theorem 2.20).

Remark 3.9. The special choices for the embeddings ϕ_{irr} and ψ (Facts 3.2.(ii) and 3.3.(ii)) will be important for the calculation of the Euler class of the Anosov reduction of the hybrid representation. Obviously one can always change one of this two embeddings by conjugation by an element of the centralizer of $\rho(\gamma)$, i.e. an element, which is of the form $\mathrm{diag}(a, b, a^{-1}, b^{-1})$, these representations are also maximal.

In order to keep track of this situation we define

Definition 3.10. Let γ be a loop⁵ on Σ and let ρ_l and ρ_r two representations of $\pi_1(\Sigma)$ into $\mathrm{Sp}(4, \mathbf{R})$ with $\rho_l(\gamma) = \rho_r(\gamma)$ and such that ρ_l is a Hitchin representation and ρ_r is a maximal representation into $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R}) < \mathrm{Sp}(4, \mathbf{R})$.

The pair (ρ_l, ρ_r) is said to be *positively adjusted with respect to γ* if there exists a symplectic basis $(\epsilon_i)_{i=1, \dots, 4}$ and continuous deformations $(\rho_{l,t})_{t \in [0,1]}$ and $(\rho_{r,t})_{t \in [0,1]}$ such that:

- $\rho_{l,1} = \rho_l$ and $\rho_{r,1} = \rho_r$,
- $\rho_{l,0} = \phi_{irr} \circ \iota$ is an irreducible Fuchsian representation, $\rho_{r,0} = \phi_\Delta \circ \iota$ is a diagonal Fuchsian representation and for each t in $[0, 1]$ $\rho_{r,t}$ is a maximal representation into $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R}) < \mathrm{Sp}(4, \mathbf{R})$,
- for each t , $\rho_{l,t}(\gamma)$ and $\rho_{r,t}(\gamma)$ are diagonal in the base (ϵ_i) .

The pair (ρ_l, ρ_r) is said to be *negatively adjusted with respect to γ* if the pair $(g\rho_l g^{-1}, \rho_r)$ is positively adjusted where g is diagonal in the base $(\epsilon_i)_{i=1, \dots, 4}$ with eigenvalues of opposite signs, i.e. $g = \mathrm{diag}(a, b, a^{-1}, b^{-1})$ with $ab < 0$.

Remark 3.11. In fact, it is not really necessary here that the representations ρ_l and ρ_r are representations of $\pi_1(\Sigma)$, since only their restrictions to $\pi_1(\Sigma_l)$ or $\pi_1(\Sigma_r)$ are considered.

Since a hybrid representation is maximal the associated flat symplectic \mathbf{R}^4 -bundle E admits a Lagrangian splitting $E = L^s(\rho) \oplus L^u(\rho)$. We are unable to describe the Lagrangian bundles $L^s(\rho)$ and $L^u(\rho)$ explicitly as we did above for standard maximal representations. Nevertheless, computing the topological invariants (see Section 5) we defined for Anosov representations, we will be able to determine the topological type of $L^s(\rho)$. The topological type will indeed only depend on the Euler characteristic of Σ_l .

⁵The loop γ does not need to be separating or simple.

3.3.2. Hybrid representations: general construction. In the construction above we decompose Σ along one simple closed separating geodesic, so the Euler characteristic of Σ_l will be odd. To obtain k -hybrid representations of $\pi_1(\Sigma)$ for all $\chi(\Sigma) + 1 \leq k \leq -1$ we have to consider slightly more general decompositions of Σ , in particular Σ_l or Σ_r might not be connected.

Let us fix some notation to describe this more general construction. Let Σ be closed oriented surface of genus g and $\Sigma_1 \subset \Sigma$ be a subsurface with Euler characteristic equal to k .

The (non-empty) boundary $\partial\Sigma_1$ is the union of disjoint circles γ_d for $d \in \pi_0(\partial\Sigma_1)$. We orient the circles so that, for each d , the subsurface Σ_1 is on the left of γ_d . Write the surface $\Sigma \setminus \partial\Sigma_1$ as the union of its connected components:

$$\Sigma \setminus \partial\Sigma_1 = \bigcup_{c \in \pi_0(\Sigma \setminus \partial\Sigma_1)} \Sigma_c.$$

For any d in $\pi_0(\partial\Sigma_1)$ the curve γ_d bounds exactly 2 connected components of $\Sigma \setminus \partial\Sigma_1$: one is included in Σ_1 and denoted by $\Sigma_{l(d)}$ with $l(d)$ in $\pi_0(\Sigma_1)$; the other one is included in the complement of Σ_1 and denoted by $\Sigma_{r(d)}$ with $r(d)$ in $\pi_0(\Sigma \setminus \Sigma_1)$. Note that $l(d)$ and $r(d)$ are elements of $\pi_0(\Sigma \setminus \partial\Sigma_1)$. In particular, $l(d)$ might equal $l(d')$ for some $d \neq d'$; similarly for $r(d)$ and $r(d')$.

We assume that

- The graph with vertex set $\pi_0(\Sigma \setminus \partial\Sigma_1)$ and edges given by the pairs $\{(l(d), r(d))\}_{d \in \pi_0(\partial\Sigma_1)}$ is a tree.

The fundamental group $\pi_1(\Sigma)$ can be described as the amalgamated product of the groups $\pi_1(\Sigma_c)$, c in $\pi_0(\Sigma \setminus \partial\Sigma_1)$, over the groups $\pi_1(\gamma_d)$, $d \in \pi_0(\partial\Sigma_1)$. The above assumption ensures that no HNN-extensions appear in this description.

With these notations, we can now define general k -hybrid representations. For each c in $\pi_0(\Sigma_1)$ we choose a representation

$$\rho_c : \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(4, \mathbf{R})$$

belonging to one of the 2^{2g} Hitchin components. We set with a slight abuse of notation $\rho_c = \rho_c|_{\pi_1(\Sigma_c)}$ for each c in $\pi_0(\Sigma_1)$.

For any d in $\pi_0(\partial\Sigma_1)$, $\rho_{l(d)}(\gamma_d)$ (note that $l(d) \in \pi_0(\Sigma_1)$) is conjugate to a unique element of the form

$$\rho_{l(d)}(\gamma_d) \cong \epsilon(d) \begin{pmatrix} e^{l_1(d)} & & & \\ & e^{l_2(d)} & & \\ & & e^{-l_1(d)} & \\ & & & e^{-l_2(d)} \end{pmatrix} \in \mathrm{Sp}(4, \mathbf{R})$$

with $\epsilon(d) \in \{\pm 1\}$, $l_1(d) > l_2(d) > 0$.

The construction of $\rho_{c'}$ for c' in $\pi_0(\Sigma \setminus \Sigma_1)$ now goes as follow. By Lemma A.4 one can choose a continuous path

$$\tau_{c',t} : \pi_1(\Sigma) \longrightarrow \mathrm{SL}(2, \mathbf{R}), \quad t \in [1, 2]$$

such that $\tau_{c',t} : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ are discrete embeddings for all $t \in [1, 2]$ and such that, for any d in $\pi_0(\partial\Sigma_{c'}) \subset \pi_0(\partial\Sigma_1)$, (hence $r(d) = c'$) one has

$$\tau_{c',i}(\gamma_d) \text{ is conjugate to } \epsilon(d) \begin{pmatrix} e^{l_i(d)} & \\ & e^{-l_i(d)} \end{pmatrix}, \text{ for } i = 1, 2.$$

Set $\rho_{c'} = \psi \circ (\tau_{c',1}, \tau_{c',2}) : \pi_1(\Sigma_{c'}) \rightarrow \mathrm{Sp}(4, \mathbf{R})$, where ψ is defined in Fact 3.3.(ii).

In order to define the amalgamated representation, we need to choose elements g_c in $\mathrm{Sp}(4, \mathbf{R})$ for any c in $\pi_0(\Sigma \setminus \partial\Sigma_1)$ such that, for any d , $g_{l(d)}\rho_{l(d)}(\gamma_d)g_{l(d)}^{-1} = g_{r(d)}\rho_{r(d)}(\gamma_d)g_{r(d)}^{-1}$. As mentioned in Remark 3.9 these elements should be chosen such that

- for any d in $\pi_0(\partial\Sigma_1)$ the pair of representations

$$(g_{l(d)}\rho_{l(d)}g_{l(d)}^{-1}, g_{r(d)}\rho_{r(d)}g_{r(d)}^{-1})$$

is positively adjusted with respect to γ_d (Definition 3.10).

Such a family (g_c) always exists by our hypothesis that the graph associated with the decomposition of the surface Σ is a tree. One then constructs the k -hybrid representation

$$\rho : \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(4, \mathbf{R})$$

by amalgamation of the representations $g_{l(d)}\rho_{l(d)}g_{l(d)}^{-1}$ and $g_{r(d)}\rho_{r(d)}(\gamma_d)g_{r(d)}^{-1}$.

Remark 3.12. The hypothesis that the dual graph of $\Sigma \setminus \partial\Sigma_1$ is a tree is necessary. For example, if this graph has a double edge, one would try to construct a Hitchin representation whose restriction to the disjoint union of two closed simple curves γ_1 and γ_2 is contained in some $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R}) < \mathrm{Sp}(4, \mathbf{R})$. However it is not difficult to see that the restriction of a Hitchin representation to the subgroup generated by γ_1 and γ_2 is irreducible and hence cannot be contained in $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R}) < \mathrm{Sp}(4, \mathbf{R})$.

3.3.3. Other amalgamated representations. Let us describe a variant of the construction of hybrid representation. Assume that Σ is decomposed along a simple closed separating geodesic γ into two subsurfaces Σ_l and Σ_r as above. On $\pi_1(\Sigma_l)$ we choose again the irreducible Fuchsian representation $\rho_{irr} = \phi_{irr} \circ \iota$ into $\mathrm{Sp}(4, \mathbf{R})$, for the fundamental group of $\pi_1(\Sigma_r)$ we choose a maximal representation into $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R}) \subset \mathrm{Sp}(4, \mathbf{R})$ which agrees with the irreducible representation along γ , but sends an element $\alpha \in \pi_1(\Sigma_r)$ corresponding to a non-separating simple closed geodesic to an element of $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$ with eigenvalues of different sign. The corresponding amalgamated representation will be maximal. However, the first Stiefel-Whitney class of this representation is non-zero by Lemma 4.11. Thus Theorem 7 implies that ρ can be deformed to a twisted diagonal representation.

Analogous constructions can be made to obtain maximal representations into other Lie groups G of Hermitian type. When G is not locally isomorphic to $\mathrm{Sp}(4, \mathbf{R})$, we expect that all maximal representations can be deformed to twisted diagonal representations.

4. TOPOLOGICAL INVARIANTS

In this section we introduce the topological invariants for Anosov representations. First the uniqueness of the *Anosov section* is restated (Section 4.1). Some general properties of obstruction classes for those Anosov sections are discussed (Section 4.2). Then we define first and second Stiefel-Whitney classes for symplectic maximal representations (Section 4.3). In Section 4.4 we focus on the special case $\mathrm{Sp}(4, \mathbf{R})$ and define an Euler class considering a cover of the representation space (*enhanced* representation space, Section 4.4.1). In Section 4.5 we prove a priori constraints on the possible values for the invariants. Finally in Section 4.6 we define and calculate invariants for other Anosov representations.

4.1. Uniqueness. Let us denote by $\text{Hom}_{H\text{-Anosov}}(\pi_1(M), G)$ the set of (G, H) -Anosov representations and let $\mathcal{B}_H(M)$ the set of gauge isomorphism classes of H -bundles over M . Summarizing Proposition 2.5 and Proposition 2.9 we have

Proposition 4.1. *For any pair (G, H) , there is a well-defined locally constant map*

$$\text{Hom}_{H\text{-Anosov}}(\pi_1(M), G) \longrightarrow \mathcal{B}_H(M),$$

associating to an Anosov representation its Anosov H -reduction.

This map is natural with respect to taking finite covers of M and with respect to the constructions described in Lemmas 2.10 and 2.11 and in Proposition 2.8.

In general $\mathcal{B}_H(M)$ could be rather complicated, so instead of the whole space of gauge isomorphism classes $\mathcal{B}_H(M)$ we will consider the obstruction classes associated with the H -bundle as topological invariants of an Anosov representation, i.e. the invariants are elements of the cohomology groups of M , possibly with local coefficients.

Remark 4.2. The group H is the Levi component of P^s , therefore P^s and H are homotopy equivalent. Thus $\mathcal{B}_H = \mathcal{B}_{P^s}$, and instead of working with the H -reduction we could equally well work with the corresponding P^s -reduction (or similarly with the P^u -reduction) of the flat G -bundle.

4.2. Characteristic classes. Obstruction theory (see [42, Part III]) associates characteristic classes to fiber bundles; they measure the obstruction to constructing sections of the bundles. In our special case, if P_H is a principal H -bundle over M we obtain a characteristic class

$$o_1(P_H) \in H^1(M; \pi_0(H)^{ab}),$$

with $\pi_0(H)^{ab}$ the abelianization of $\pi_0(H)$. This class can be explicitly described as follows: the associated principal $\pi_0(H)$ -bundle $P_H \times_H \pi_0(H)$ is necessarily flat (since its structure group is discrete) and hence comes from a morphism $\pi_1(M) \rightarrow \pi_0(H)$. This latter morphism defines $o_1(P_H)$ a class in $H^1(M; \pi_0(H)^{ab}) = H^1(\pi_1(M); \pi_0(H)^{ab}) \cong \text{Hom}(\pi_1(M), \pi_0(H)^{ab})$.

Definition 4.3. The *first obstruction class* $o_1(\rho)$ of a (G, H) -Anosov representation $\rho : \pi_1(M) \rightarrow G$ is the class $o_1(P_H)$ of the Anosov reduction P_H .

Remark 4.4. A more precise invariant would be the first obstruction morphism $\sigma_1(\rho)$ in $\text{Rep}(\pi_1(M), \pi_0(H))$. In every case considered below $\pi_0(H)$ is abelian, so that the obstruction morphism and the obstruction class are equal. This is not, however, the general case.

In order to get a characteristic class of degree $q + 1$ one need to consider bundles whose fibers are q -connected ([42, Sec. 29]). We restrict here the discussion to $q = 1$. A way to get a class in a second cohomology group is to consider the H/F -bundle P_H/F where F is a subgroup of H such that H/F is connected. We will define a second cohomology class under the following

Condition 4.5. The group H is the semidirect product of its identity component H_0 and a discrete group F ; $H = F \ltimes H_0$.

In that situation $F \cong \pi_0(H)$ and $\pi_1(H/F) \cong \pi_1(H_0)$. The obstruction $o_2(P_H)$ to finding a section of the bundle P_H/F is a class in $H^2(M; \pi_1(H/F))$ the second

cohomology group of M with coefficients in the $\pi_1(M)$ -module $\pi_1(H/F)$ (for generalities on cohomology with local coefficients see [28, Sec. 3.H] or [42, Sec. 31]). The action of $\pi_1(M)$ on $\pi_1(H/F)$ is the composition $\pi_1(M) \rightarrow \pi_0(H) \rightarrow \text{Aut}(\pi_1(H/F))$.

Here is a way to construct this second obstruction class. Using Condition 4.5 it is easy to construct a (noncentral) extension $\pi_1(H_0) \rightarrow \overline{H} \rightarrow H$. The obstruction to write the H -bundle P_H as the reduction of a \overline{H} -bundle is a class in $H^2(M; \pi_1(H_0))$ (cohomology with local coefficients since the extension is not central) and is the second obstruction class $o_2(\rho)$.

The next two results describe the behavior of the first obstruction under covering and when twisting by a representation into the center.

Proposition 4.6. *Let $\pi : G' \rightarrow G$ be a finite covering of Lie groups. Suppose that a representation $\rho : \pi_1(M) \rightarrow G'$ is H' -Anosov.*

Then the representation $\pi \circ \rho$ is $H = \pi(H')$ -Anosov and under the natural map

$$H^1(M, \pi_0(H')) \rightarrow H^1(M, \pi_0(H))$$

the class $o_1(\pi \circ \rho)$ is the image of $o_1(\rho)$.

Proof. By Lemma 2.11 $\pi \circ \rho$ is H -Anosov with Anosov reduction $P_{H'} \times_{H'} H$ where $P_{H'}$ is the Anosov reduction of ρ . Hence the morphism $\pi_1(M) \rightarrow \pi_0(H)^{ab}$ defining $o_1(\pi \circ \rho)$ is the composition $\pi_1(M) \rightarrow \pi_0(H')^{ab} \rightarrow \pi_0(H)^{ab}$ of $o_1(\rho)$ with the natural projection $\pi_0(H')^{ab} \rightarrow \pi_0(H)^{ab}$. \square

Proposition 4.7. *Let $\rho : \pi_1(M) \rightarrow G$ be a (G, H) -Anosov representation. Let Z be the center of G and $\eta : \pi_1(M) \rightarrow Z$ a representation.*

Then the representation $\eta \cdot \rho : \pi_1(M) \rightarrow G$, $\gamma \mapsto \eta(\gamma)\rho(\gamma)$ is (G, H) -Anosov and its first obstruction class is

$$o_1(\eta \cdot \rho) = o_1(\rho) + \overline{\eta}$$

with $\overline{\eta}$ being the composition $\pi_1(M) \rightarrow Z \rightarrow \pi_0(H)^{ab}$.

Proof. The representation $(\eta, \rho) : \pi_1(M) \rightarrow Z \times G$ is Anosov with Anosov reduction $P_\eta \times P_H$, where P_H is the Anosov reduction of ρ and P_η is the flat Z -bundle associated with η . Hence, by Lemma 2.11, the representation $\eta \cdot \rho$ is Anosov with reduction $(P_\eta \times P_H)/Z$. Thus the obstruction morphism $\sigma_1(\eta \cdot \rho)$ (see Remark 4.4) is the composition of $(\eta, \sigma_1(\rho))$ with the map $Z \times \pi_0(H) \rightarrow \pi_0(H)$. The result for the obstruction class follows from this description. \square

A direct application of these propositions to the study of connected components is the following result. To allow more flexibility we will work with a Galois covering $\overline{M} \rightarrow M$ with group Γ and consider representations of Γ . The group Γ is a quotient of $\pi_1(M)$.

Proposition 4.8. *Let $\pi : G' \rightarrow G$ be a finite covering of connected Lie groups with kernel W . Let $H' < G'$ be the intersection of two opposite parabolic subgroups and $H = \pi(H')$.*

Let $p : \text{Hom}(\Gamma, G') \rightarrow \text{Hom}(\Gamma, G)$ be the map $\rho \mapsto \pi \circ \rho$ and let \mathcal{C} be a connected subset of $\text{Hom}_{H\text{-Anosov}}(\Gamma, G)$.

Suppose that the natural map $H^1(\Gamma; W) \cong \text{Hom}(\Gamma, W) \rightarrow H^1(M; \pi_0(H')^{ab}) \cong \text{Hom}(\pi_1(M), \pi_0(H')^{ab})$ is an injection then

– either $p^{-1}(\mathcal{C})$ is empty,

- or $p^{-1}(\mathcal{C})$ is the union of $|\mathrm{H}^1(\Gamma; W)|$ components, that are distinguished by the first obstruction o_1 . The image of $o_1 : p^{-1}(\mathcal{C}) \rightarrow \mathrm{H}^1(M; \pi_0(H')^{ab})$ is one coset for the subgroup $\mathrm{H}^1(\Gamma; W)$.

Proof. If $p^{-1}(\mathcal{C})$ is nonempty then by connectedness every representation in \mathcal{C} lifts to $\mathrm{Hom}(\Gamma, G')$. Thus $p^{-1}(\mathcal{C}) \rightarrow \mathcal{C}$ is a finite covering. Note that the fiber containing a representation ρ is precisely the set $\eta \cdot \rho$ for $\eta : \Gamma \rightarrow W$. Hence this covering is of degree $|\mathrm{H}^1(\Gamma; W)|$. So $p^{-1}(\mathcal{C})$ has at most that number of components.

Note also that $p^{-1}(\mathcal{C})$ is included in $\mathrm{Hom}_{H' \text{-Anosov}}(\Gamma, G')$. Therefore the obstruction class defines a map

$$o_1 : p^{-1}(\mathcal{C}) \longrightarrow \mathrm{H}^1(M; \pi_0(H')^{ab}).$$

Using Proposition 4.7, for any ρ in $p^{-1}(\mathcal{C})$ and any η in $\mathrm{H}^1(\Gamma; W)$ the two classes $o_1(\rho)$ and $o_1(\eta \cdot \rho)$ differ by the image of η in $\mathrm{H}^1(M; \pi_0(H')^{ab})$. The image of o_1 is hence one orbit of $\mathrm{H}^1(\Gamma; W)$ in $\mathrm{H}^1(M; \pi_0(H')^{ab})$. The hypothesis that $\mathrm{H}^1(\Gamma; W)$ injects into $\mathrm{H}^1(M; \pi_0(H')^{ab})$ insures then that the image of o_1 has at least $|\mathrm{H}^1(\Gamma; W)|$ elements and hence $p^{-1}(\mathcal{C})$ has at least that number of components. \square

The next lemma will provide an easy criterion for the hypothesis of the above proposition to be satisfied.

Lemma 4.9. *Let Σ be a closed oriented connected surface. Suppose that $A \rightarrow B$ is an injective morphism of abelian group. Then the map*

$$\mathrm{H}^1(\Sigma; A) \longrightarrow \mathrm{H}^1(T^1\Sigma; B)$$

is injective.

Proof. The map between the fundamental groups $\pi_1(T^1\Sigma) \rightarrow \pi_1(\Sigma)$ is surjective. Together with the injectivity of $A \rightarrow B$ this implies the injectivity of $\mathrm{H}^1(\Sigma; A) \cong \mathrm{Hom}(\pi_1(\Sigma), A) \rightarrow \mathrm{H}^1(T^1\Sigma; B) \cong \mathrm{Hom}(\pi_1(T^1\Sigma), B)$. \square

4.3. First and second Stiefel-Whitney classes. The inclusion

$$\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) \subset \mathrm{Hom}_{\mathrm{GL}(n, \mathbf{R})\text{-Anosov}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})),$$

which is described in Section 2.2.2, allows us to apply Proposition 4.1 to associate to a maximal representation into $\mathrm{Sp}(2n, \mathbf{R})$ the first and second Stiefel-Whitney classes of a $\mathrm{GL}(n, \mathbf{R})$ -bundle over $T^1\Sigma$.

Proposition 4.10. *Let $G = \mathrm{Sp}(2n, \mathbf{R})$ and $\mathrm{Hom}_{\max}(\pi_1(\Sigma), G)$ the space of maximal representations. Then the obstruction classes of the Anosov $\mathrm{GL}(n, \mathbf{R})$ -reduction give maps:*

$$\begin{aligned} \mathrm{Hom}_{\max}(\pi_1(\Sigma), G) &\xrightarrow{sw_1} \mathrm{H}^1(T^1\Sigma; \mathbf{F}_2) \\ \text{and } \mathrm{Hom}_{\max}(\pi_1(\Sigma), G) &\xrightarrow{sw_2} \mathrm{H}^2(T^1\Sigma; \mathbf{F}_2). \end{aligned}$$

The following geometric interpretation makes the first Stiefel-Whitney class $sw_1(\rho)$ easy to compute. Recall that given a smooth closed curve γ on Σ , taking the velocity vector at every point defines a natural lift to a loop on $T^1\Sigma$; this gives rise to a natural map $\pi_1(\Sigma) \setminus \{1\} \rightarrow \mathrm{H}_1(T^1\Sigma; \mathbf{Z})$, $\gamma \mapsto [\gamma]$.

Lemma 4.11. *Let $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ be a maximal representation and $\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{L}$ the equivariant limit curve. Then*

$$sw_1(\rho)([\gamma]) = \mathrm{sign}(\det \rho(\gamma)|_{\xi(t_\gamma^s)}),$$

where $t_\gamma^s \in \partial\pi_1(\Sigma)$ is the attractive fixed point of γ and \mathbf{F}_2 is identified with $\{\pm 1\}$.

Proof. The first Stiefel-Whitney class $sw_1(\rho)([\gamma])$ is the obstruction to the orientability of the bundle $L^s(\rho)|_\gamma \cong \mathbf{Z} \backslash (\mathbf{R} \times \xi(t_\gamma^s))$ over $\gamma \cong \mathbf{Z} \backslash \mathbf{R}$. Thus, $sw_1(\rho)([\gamma]) = 1$ if $\rho(\gamma) \in \mathrm{GL}(\xi(t_\gamma^s))$ has positive determinant and $sw_1(\rho)([\gamma]) = -1$ otherwise. \square

4.4. An Euler class. For $n = 2$, the invariants obtained in Proposition 4.10 do not allow to distinguish the connected components of

$$\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \setminus \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})).$$

In particular, the second Stiefel-Whitney class does not offer enough information to distinguish the connected components of

$$\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})).$$

The reason for this is that when $sw_1(\rho) = 0$ the Lagrangian bundle $L^s(\rho)$ is orientable. There is an Euler class whose image in $H^2(T^1\Sigma; \mathbf{F}_2)$ is the second Stiefel-Whitney class $sw_2(\rho)$. Since an *orientable* vector bundle has no *canonical orientation*, we need to introduce an *enhanced* representation space to obtain a well-defined Euler class.

4.4.1. Enhanced representation spaces. Let $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ be a maximal representation. Let $\gamma \in \pi_1(\Sigma) \setminus \{1\}$ and $L^s(\gamma) \in \mathcal{L}$ the attractive fixed Lagrangian of $\rho(\gamma)$. We denote by \mathcal{L}_+ the space of *oriented* Lagrangians and by $\pi : \mathcal{L}_+ \rightarrow \mathcal{L}$ the projection. Let

$$\begin{aligned} \mathrm{Hom}_{\max, sw_1=0}^{\mathcal{L}, \gamma}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) = \\ \{(\rho, L) \in \mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \times \mathcal{L} \mid L = L^s(\gamma)\} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) = \\ \{(\rho, L_+) \in \mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \times \mathcal{L}_+ \mid \pi(L_+) = L^s(\gamma)\}. \end{aligned}$$

The map $\mathrm{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \rightarrow \mathrm{Hom}_{\max, sw_1=0}^{\mathcal{L}, \gamma}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ is a 2-fold cover.

The space $\mathrm{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ is easily identified with the space $\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$, we introduce it to emphasize the fact that $\rho(\gamma)$ has an attractive Lagrangian.

Lemma 4.12. *The natural map*

$$\begin{aligned} \mathrm{Hom}_{\max, sw_1=0}^{\mathcal{L}, \gamma}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) &\longrightarrow \mathcal{L} \\ (\rho, L) &\longmapsto L \end{aligned}$$

is continuous.

This lemma follows immediately from the continuity of the eigenspace of a matrix or from Proposition 2.9.

4.4.2. The Euler class. As a consequence of the following proposition for every element (ρ, L_+) in $\mathrm{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma}$ there is a natural associated *oriented* Lagrangian bundle over $T^1\Sigma$.

Proposition 4.13. *Let $\rho \in \text{Hom}_{\max}(\pi_1(\Sigma), \text{Sp}(2n, \mathbf{R}))$ and let $\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{L}$ be the corresponding equivariant positive curve. Suppose that n is even. Then there exists a continuous lift of ξ to \mathcal{L}_+ .*

Let $\xi_+ : \partial\pi_1(\Sigma) \rightarrow \mathcal{L}_+$ be one of the two continuous lifts of ξ . Then the map ξ_+ is ρ -equivariant if and only if $sw_1(\rho) = 0$. In this case the other lift of ξ is also equivariant.

Proof. The fact that a continuous lift exists depends only on the homotopy class of the curve ξ . Since the space of continuous and positive curves is connected (see Proposition A.1), the existence of a lift can be checked for one specific maximal representation. Considering a Fuchsian representation gives the restriction on n .

Let ξ_+ be a continuous lift of ξ and denote by ξ'_+ the other lift of the curve ξ . Since ξ is equivariant, for any γ in $\pi_1(\Sigma)$, the following alternative holds:

$$\gamma \cdot \xi_+ = \xi_+ \text{ or } \gamma \cdot \xi_+ = \xi'_+.$$

Furthermore, given any t in $\partial\pi_1(\Sigma)$, by connectedness one observes that

$$\gamma \cdot \xi_+ = \xi_+ \iff \xi_+(\gamma \cdot t) = \rho(\gamma) \cdot \xi_+(t).$$

Using this last equation for the attractive fixed point t_γ^s of γ in $\partial\pi_1(\Sigma)$, the equivalence becomes

$$\gamma \cdot \xi_+ = \xi_+ \iff \det \rho(\gamma)|_{\xi(t_\gamma^s)} > 0.$$

Using now the equality $sw_1(\rho)([\gamma]) = \text{sign}(\det \rho(\gamma)|_{\xi(t_\gamma^s)})$ (Lemma 4.11), the proposition follows. \square

Proposition 4.13 gives a natural way to lift the equivariant curve given any element $(\rho, L_+) \in \text{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$:

Definition 4.14. Let $(\rho, L_+) \in \text{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$ and let $\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{L}$ be the ρ -equivariant curve. The lift (uniquely determined and equivariant) ξ_+ of ξ such that

$$\xi_+(t_\gamma^s) = L_+$$

is called the *canonical oriented equivariant curve* for the pair (ρ, L_+) . Here $t_\gamma^s \in \partial\pi_1(\Sigma)$ is the attractive fixed point of γ .

This canonical oriented equivariant curve defines a $\text{GL}^+(2, \mathbf{R})$ -reduction of the $\text{GL}(2, \mathbf{R})$ -Anosov reduction associated with ρ . It also induces a splitting of the flat bundle E associated with ρ

$$E = L_+^s(\rho) \oplus L_+^u(\rho)$$

into two oriented Lagrangian subbundles. We call this splitting the *oriented Lagrangian reduction* of E .

Definition 4.15. The Euler class

$$\begin{aligned} e_\gamma : \text{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) &\longrightarrow \text{H}^2(T^1\Sigma; \mathbf{Z}) \\ (\rho, L_+) &\longmapsto e_\gamma(\rho, L_+) \end{aligned}$$

is the Euler class of the $\text{GL}^+(2, \mathbf{R})$ -reduction given by Definition 4.14.

The map e_γ is continuous.

4.4.3. *Connected components.* We consider now a subspace of $\text{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma}$ by fixing the oriented Lagrangian. Let $L_{0+} \in \mathcal{L}_+$ be an oriented Lagrangian; we set

$$\begin{aligned} \text{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma, L_{0+}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) = \\ \{(\rho, L_+) \in \text{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \mid L_+ = L_{0+}\}. \end{aligned}$$

Lemma 4.16. *The natural map*

$$\text{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma, L_{0+}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \longrightarrow \text{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) / \text{Sp}(4, \mathbf{R})$$

is onto. Its fibers are connected.

Proof. Let $\rho \in \text{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$. Since the first Stiefel-Whitney class of ρ vanishes, there exists an attracting oriented Lagrangian L_+ for $\rho(\gamma)$. Let $g \in \text{Sp}(4, \mathbf{R})$ such that $g \cdot L_+ = L_{0+}$, then $(g\rho g^{-1}, L_{0+})$ belongs to $\text{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma, L_{0+}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$ and projects to $[\rho]$.

Now, let $(\rho, L_{0+}) \in \text{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma, L_{0+}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$. Then the fiber of the projection containing (ρ, L_{0+}) is isomorphic to the quotient:

$$\{g \in \text{Sp}(4, \mathbf{R}) \mid g \cdot L_{0+} = L_{0+}\} / \{z \in Z(\rho) \mid z \cdot L_{0+} = L_{0+}\},$$

where $Z(\rho)$ denotes the centralizer of $\rho(\pi_1(\Sigma))$ in $\text{Sp}(4, \mathbf{R})$. The group $\{g \in \text{Sp}(4, \mathbf{R}) \mid g \cdot L_{0+} = L_{0+}\}$ is connected. Thus the fiber is connected. \square

Lemma 4.16 implies that $\text{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma, L_{0+}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$ has as the same number of connected components as $\text{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$.

Definition 4.17. Let $\gamma \in \pi_1(\Sigma) \setminus \{1\}$ and $L_{0+} \in \mathcal{L}_+$. The *relative Euler class* $e_{\gamma, L_{0+}}(\rho) \in H^2(T^1\Sigma; \mathbf{Z})$ of the class of a maximal representation $\rho \in \text{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$ is defined as the Euler class of one (any) inverse image of $[\rho]$ in the space $\text{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma, L_{0+}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$.

4.5. **Constraints on invariants.** Since the invariants are constructed from bundles that are flat along geodesic leaves, one expects that not every cohomology class can arise:

Proposition 4.18. *Let $G = \text{Sp}(2n, \mathbf{R})$. Then*

(i) *The image of $sw_1 : \text{Hom}_{\max}(\pi_1(\Sigma), G) \rightarrow H^1(T^1\Sigma; \mathbf{F}_2)$ is contained in one coset of $H^1(\Sigma; \mathbf{F}_2)$. More precisely:*

- *for n even, $sw_1(\rho) \in H^1(\Sigma; \mathbf{F}_2) \subset H^1(T^1\Sigma; \mathbf{F}_2)$,*
- *for n odd, $sw_1(\rho) \in H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$.*

(ii) *The image of $sw_2 : \text{Hom}_{\max}(\pi_1(\Sigma), G) \rightarrow H^2(T^1\Sigma; \mathbf{F}_2)$ lies in the image of $H^2(\Sigma; \mathbf{F}_2) \rightarrow H^2(T^1\Sigma; \mathbf{F}_2)$.*

(iii) *Similarly, when $n = 2$, let $\gamma \in \pi_1(\Sigma)$ and $L_{0+} \in \mathcal{L}_+$, then the image of $e_{\gamma, L_{0+}} : \text{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \rightarrow H^2(T^1\Sigma; \mathbf{Z})$ lies in the image of $H^2(\Sigma; \mathbf{Z}) \rightarrow H^2(T^1\Sigma; \mathbf{Z})$.*

To prove this proposition we use the positivity of the equivariant curve $\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{L}$ of a maximal representation $\rho : \pi_1(\Sigma) \rightarrow \text{Sp}(2n, \mathbf{R})$ (Definition 2.24).

Proof. For the first property, let $x \in \Sigma$ and consider the exact sequence

$$H^1(\Sigma; \mathbf{F}_2) \longrightarrow H^1(T^1\Sigma; \mathbf{F}_2) \xrightarrow{f_x} H^1(T_x^1\Sigma; \mathbf{F}_2).$$

We need to check that the image of $sw_1(\rho)$ in $H^1(T_x^1\Sigma; \mathbf{F}_2) \cong \mathbf{F}_2$ does not depend on ρ , i.e. that the gauge isomorphism class of $L^s(\rho)|_{T_x^1\Sigma}$ is independent of ρ .

Let $\tilde{x} \in \tilde{\Sigma}$ be a lift of x so that $T_x^1\Sigma \cong T_{\tilde{x}}^1\tilde{\Sigma} \cong \partial\pi_1(\Sigma)$, where the last map is given by the restriction of $\partial\pi_1(\Sigma)^{(3+)} \rightarrow \partial\pi_1(\Sigma)$, $(t^s, t, t^u) \mapsto t^s$ to $T_{\tilde{x}}^1\tilde{\Sigma} \subset T^1\tilde{\Sigma} \cong \partial\pi_1(\Sigma)^{(3+)}$. The restriction of the flat G -bundle to $T_x^1\Sigma$ is trivial, thus the restriction of the Anosov section σ to $T_x^1\Sigma$ can be regarded as a map $T_x^1\Sigma \rightarrow \mathcal{X}$.

By Remark 4.2, we can work with the P^s -reduction, i.e. we consider only the first component of the map $\sigma|_{T_x^1\Sigma} : T_x^1\Sigma \rightarrow \mathcal{L} \times \mathcal{L}$. With the above identification $T_x^1\Sigma \cong \partial\pi_1(\Sigma)$ this map is exactly the equivariant limit curve $\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{L}$. Since the space of positive curves is connected (Proposition A.1), $f_x(sw_1(\rho))$ is independent of ρ . The calculation for the diagonal Fuchsian representation (see Section 5.2) gives the desired statement about $sw_1(\rho)$.

For the second statement, in view of Lemma B.5 it is sufficient to prove that for any closed curve $\eta \subset \Sigma$ the restriction of $sw_2(\rho)$ (or $e_{\gamma, L_{0+}}(\rho)$) to the torus $T^1\Sigma|_\eta$ is zero.

For this we write the torus $T^1\Sigma|_\eta$ as the quotient of $\partial\pi_1(\Sigma) \times \mathbf{R}$ by $\langle \eta \rangle \cong \mathbf{Z}$ where $\eta \cdot (\theta, t) = (\eta \cdot \theta, t+1)$. With this identification the flat bundle can be written as $\langle \gamma \rangle \backslash (\partial\pi_1(\Sigma) \times \mathbf{R} \times G)$ with $\gamma \cdot (\theta, t, g) = (\gamma \cdot \theta, t+1, Ag)$ and $A \in H \cong \mathrm{GL}(n, \mathbf{R})$. The section of the associated \mathcal{L} -bundle lifts to $\partial\pi_1(\Sigma) \times \mathbf{R}$:

$$\begin{aligned} \partial\pi_1(\Sigma) \times \mathbf{R} &\longrightarrow \mathcal{L} \\ (\theta, t) &\longmapsto \xi(\theta). \end{aligned}$$

Since the space of pairs

$$\{(A, \xi) \mid A \in H, \xi \text{ positive, continuous and } A\text{-equivariant}\}$$

has exactly two connected components given by the sign of $\det A$ (Proposition A.2), we conclude that $sw_2(L^s(\rho)|_{T^1\Sigma|_\eta})$ depends only on this sign, hence $sw_2(\rho)$ depends only on $sw_1(\rho)$ (see Lemma 4.11). Calculations for the model representations show that in fact $sw_2(L^s(\rho)|_{T^1\Sigma|_\eta})$ is always zero.

The proof for the Euler class proceeds along the same lines. \square

In view of Proposition 4.18 we make the following

Definition 4.19. A pair

$$(\alpha, \beta) \in H^1(T^1\Sigma; \mathbf{F}_2) \times H^2(T^1\Sigma; \mathbf{F}_2)$$

is called *n-admissible* if

- n is even and α lies in the image of $H^1(\Sigma; \mathbf{F}_2) \rightarrow H^1(T^1\Sigma; \mathbf{F}_2)$ and β lies in the image of $H^2(\Sigma; \mathbf{F}_2) \rightarrow H^2(T^1\Sigma; \mathbf{F}_2)$.
- n is odd and α lies in $H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$, and β lies in the image of $H^2(\Sigma; \mathbf{F}_2) \rightarrow H^2(T^1\Sigma; \mathbf{F}_2)$.

For $n = 2$, a class $\beta \in H^2(T^1\Sigma; \mathbf{Z})$ is called 2-admissible if β lies in the image of $H^2(\Sigma; \mathbf{Z}) \rightarrow H^2(T^1\Sigma; \mathbf{Z})$.

4.6. Invariants for other Anosov representations.

4.6.1. *Maximal representations into $\mathrm{SL}(2, \mathbf{R})$.* Let $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R}) = \mathrm{Sp}(2, \mathbf{R})$ be a maximal representation. The Anosov section associated with ρ gives a reduction of the structure group of the flat $\mathrm{SL}(2, \mathbf{R})$ -principal bundle \mathbf{P} over $T^1\Sigma$ to $\mathrm{GL}(1, \mathbf{R})$. Considering the \mathbf{R}^2 -bundle E associated with \mathbf{P} , this reduction corresponds to a Lagrangian subbundle L of E . As invariants we get the first Stiefel-Whitney class of the line bundle L over $T^1\Sigma$,

$$sw_1(\iota) \in H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2).$$

This first Stiefel-Whitney class is precisely the spin structure on Σ which corresponds to the chosen lift of $\pi_1(\Sigma) \subset \mathrm{PSL}(2, \mathbf{R})$ to $\mathrm{SL}(2, \mathbf{R})$. The invariant $sw_1(\iota)$ can take 2^{2g} different values, distinguishing the 2^{2g} connected components of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{SL}(2, \mathbf{R}))$.

4.6.2. *The invariants for Hitchin representations into $\mathrm{Sp}(2n, \mathbf{R})$.* Let $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ be a Hitchin representation. Then ρ is indeed a $(\mathrm{Sp}(2n, \mathbf{R}), A)$ -Anosov representation, where A is the subgroup of diagonal matrices [36, Theorems 4.1, 4.2]. The reduction of the structure group of the flat $\mathrm{Sp}(2n, \mathbf{R})$ -principal bundle \mathbf{P} over $T^1\Sigma$ to A corresponds to a splitting of the associated \mathbf{R}^{2n} -bundle E into the sum of $2n$ isomorphic line bundles $F_1 \oplus \cdots \oplus F_{2n}$. The first Stiefel-Whitney class of the line bundle F_1 gives an invariant

$$sw_1^A(\rho) \in H^1(T^1\Sigma; \mathbf{F}_2),$$

which lies in $H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$. This invariant distinguishes the 2^{2g} components of $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$.

4.6.3. *General Hitchin components.* Let G^{ad} a adjoint split real simple Lie group. The principal subgroup is a distinguished $\sigma : \mathrm{PSL}(2, \mathbf{R}) \rightarrow G^{ad}$ that generalizes the n -dimensional irreducible $\mathrm{PSL}(2, \mathbf{R})$ in $\mathrm{PSL}(n, \mathbf{R})$. The Hitchin component $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), G^{ad})$ for G^{ad} is the component of $\mathrm{Hom}(\pi_1(\Sigma), G^{ad})$ that contains uniformizations $\pi_1(\Sigma) \rightarrow \mathrm{PSL}(2, \mathbf{R}) \xrightarrow{\sigma} G^{ad}$. We refer to Hitchin's paper [31] for details. Representations in $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), G^{ad})$ are called Hitchin representations.

Applying the result [16, Theorem 1.15] one obtains that Hitchin representations are H -Anosov where H is the Levi component of the Borel subgroup. In fact (by connectedness) the Anosov reduction P_H always admit a reduction to the finite index subgroup H_0 . Extending slightly the terminology we will say that the representations are H_0 -Anosov.

Now if G is a connected cover of G^{ad} so that the center Z of G is a quotient of $\pi_1(G^{ad})$: $Z = \pi_1(G^{ad})/\pi_1(G)$, one can ask when Hitchin representations lift and how many components there are above the Hitchin component $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), G^{ad})$. For this let us denote $\sigma_* : \mathbf{Z} \cong \pi_1(\mathrm{PSL}(2, \mathbf{R})) \rightarrow \pi_1(G^{ad})$ the morphism induced by the injection of principal $\mathrm{PSL}(2, \mathbf{R})$. Applying Propositions 4.8 and 4.7 one gets

Theorem 4.20. *Let $\pi : G \rightarrow G^{ad}$ be a finite connected covering of a split real adjoint simple Lie group with $\ker \pi = Z$ and let $p : \mathrm{Hom}(\pi_1(\Sigma), G) \rightarrow \mathrm{Hom}(\pi_1(\Sigma), G^{ad})$ the induced map. The group $\pi^{-1}(H_0)$ is (isomorphic to) the product $ZH_0 = Z \times H_0$.*

The Hitchin component \mathcal{H} of G^{ad} lifts if and only if $\sigma_(\chi(\Sigma))$ is in $\pi_1(G) < \pi_1(G^{ad})$. Furthermore, in that case, $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), G) = p^{-1}(\mathcal{H})$ is included in*

the space $\text{Hom}_{ZH_0\text{-Anosov}}(\pi_1(\Sigma), G)$ and the map

$$o_1 : \text{Hom}_{\text{Hitchin}}(\pi_1(\Sigma), G) \longrightarrow H^1(T^1\Sigma; \pi_0(ZH_0)) = H^1(T^1\Sigma; Z)$$

induces a bijection between the components of $\text{Hom}_{\text{Hitchin}}(\pi_1(\Sigma), G)$ and one orbit of $H^1(\Sigma; Z)$. This orbit is the inverse image of $\sigma_*(1)$ by the map $H^1(T^1\Sigma; Z) \rightarrow H^1(T_x^1\Sigma; Z) \cong Z$. In particular there are $|Z|^{2g}$ components.

Proof. The obstruction to lifting $\rho : \pi_1(\Sigma) \rightarrow G^{ad}$ to the cover $G \rightarrow G^{ad}$ lies in $H^2(\Sigma; \pi_1(G^{ad})/\pi_1(G)) \cong \pi_1(G^{ad})/\pi_1(G)$ and is the image of $o(\rho) \in \pi_1(G^{ad})$ the obstruction to lifting ρ to the universal cover of G^{ad} . For a representation $\pi_1(\Sigma) \xrightarrow{\iota} \text{PSL}(2, \mathbf{R}) \xrightarrow{\sigma} G^{ad}$ the obstruction class $o(\sigma \circ \iota)$ is the image by $\sigma_* : \mathbf{Z} \rightarrow \pi_1(G^{ad})$ of the obstruction class $o(\iota) = \chi(\Sigma)$, i.e. $o(\sigma \circ \iota) = \sigma_*(\chi(\Sigma))$. This gives the condition on $\sigma_*(\chi(\Sigma))$.

By Proposition 4.8 and Lemma 4.9 the image of o_1 in $H^1(T^1\Sigma; Z)$ is one coset of the subgroup $H^1(\Sigma; Z)$. To determine this coset observe first that the sequence $0 \rightarrow H^1(\Sigma; Z) \rightarrow H^1(T^1\Sigma; Z) \xrightarrow{i^*} H^1(T_x^1\Sigma; Z) \rightarrow 0$ is exact, where $i : T_x^1\Sigma \rightarrow T^1\Sigma$ is the injection; this follows by example from the fact that the extension $\pi_1(T^1\Sigma) \rightarrow \pi_1(\Sigma)$ is central with kernel generated by $\pi_1(T_x^1\Sigma)$. Therefore it is enough to determine $i^*(o_1(\rho))$ for one particular representation ρ . Let thus chose a representation $\rho : \pi_1(\Sigma) \rightarrow G$ lifting a representation of the form $\sigma \circ \iota$ where $\iota : \pi_1(\Sigma) \rightarrow \text{PSL}(2, \mathbf{R})$ is a discrete embedding. The unit tangent bundle $T^1\Sigma$ is then identified with $\iota(\pi_1(\Sigma)) \backslash \text{PSL}(2, \mathbf{R})$. The Anosov reduction associated with $\sigma \circ \iota$ is the equivariant map

$$\begin{aligned} \text{PSL}(2, \mathbf{R}) &\longrightarrow G^{ad}/H_0 \\ g &\longmapsto \sigma(g)H_0, \end{aligned}$$

and the Anosov reduction for ρ is

$$\begin{aligned} \text{PSL}(2, \mathbf{R}) &\longrightarrow G/ZH_0 \\ g &\longmapsto \pi^{-1}(\sigma(g)H_0). \end{aligned}$$

The image of $o_1(\rho)$ in $H^1(T_x^1\Sigma; Z)$ is the holonomy of the corresponding principal $\pi_0(ZH_0) \cong Z$ -bundle restricted to $T_x^1\Sigma \cong \text{PSO}(2) \subset \text{PSL}(2, \mathbf{R})$. It is calculated as follow: Let $(g_t)_{t \in [0,1]}$ a loop generating $\pi_1(\text{PSO}(2)) = \pi_1(\text{PSL}(2, \mathbf{R}))$ and let $(h_t)_{t \in [0,1]}$ be a continuous curve such that, for all t , h_t belongs to $\pi^{-1}(\sigma(g_t)H_0)$. Then $h_1h_0^{-1}$ is in ZH_0 and the sought for holonomy is the projection onto Z of this element. However one can chose $h_t \in \pi^{-1}(\sigma(g_t))$. For this choice $h_1h_0^{-1}$ is in Z and is the image of the loop $(\sigma(g_t))_{t \in [0,1]}$ under the natural map $\pi_1(G^{ad}) \rightarrow Z$. Since the loop $(\sigma(g_t))$ represents precisely $\sigma_*(1)$, the result follows. \square

For classical split Lie groups the obstruction class obtained is the first Stiefel-Whitney class sw_1 of a line bundle. We have:

Theorem 4.21. *For G the group $\text{SL}(n, \mathbf{R})$, $\text{Sp}(n, \mathbf{R})$, $\text{SO}_o(n, n)$ or $\text{SO}_o(n, n+1)$, the map*

$$sw_1 : \text{Hom}_{\text{Hitchin}}(\pi_1(\Sigma), G) \longrightarrow H^1(T^1\Sigma, \mathbf{F}_2)$$

induces a bijection between $\pi_0(\text{Hom}_{\text{Hitchin}})$ and $H^1(T^1\Sigma, \mathbf{F}_2) \setminus H^1(\Sigma, \mathbf{F}_2)$.

Furthermore, for any ρ in $\text{Hom}_{\text{Hitchin}}$ and any nontrivial γ in $\pi_1(\Sigma)$, $sw_1(\rho)([\gamma])$ in $\mathbf{F}_2 \cong \{\pm 1\}$ is the common sign of the eigenvalues of $\rho(\gamma)$.

4.6.4. *Maximal representations into covers of $\mathrm{Sp}(2n, \mathbf{R})$.* The discussion of Section 4.2 can be used to analyze maximal representations into covers of $\mathrm{Sp}(2n, \mathbf{R})$.

Let $\mathrm{Sp}(2n, \mathbf{R})_{(k)}$ be the k -fold connected cover of $\mathrm{Sp}(2n, \mathbf{R})$, i.e. it corresponds to the subgroup $k\mathbf{Z} \subset \mathbf{Z} \cong \pi_1(\mathrm{Sp}(2n, \mathbf{R}))$. The kernel Z_k of the projection $\pi : \mathrm{Sp}(2n, \mathbf{R})_{(k)} \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ is isomorphic to $\mathbf{Z}/k\mathbf{Z}$. The subgroups $H_{(k)} = \pi^{-1}(\mathrm{GL}(n, \mathbf{R}))$ and $H_{(k)}^+ = \pi^{-1}(\mathrm{GL}^+(n, \mathbf{R}))$ are k -fold covers of $\mathrm{GL}(n, \mathbf{R})$ and $\mathrm{GL}^+(n, \mathbf{R})$.

Lemma 4.22. *The covers $H_{(k)}^+ \rightarrow \mathrm{GL}^+(n, \mathbf{R})$ and $H_{(k)} \rightarrow \mathrm{GL}(n, \mathbf{R})$ are trivial. The natural map*

$$Z_k \longrightarrow \pi_0(H_{(k)}^+)$$

is an isomorphism. The map $Z_k \rightarrow \pi_0(H_{(k)})$ is injective with image a subgroup of index 2. The group $\pi_0(H_{(k)})$ is isomorphic to $\mathbf{Z}/2k\mathbf{Z}$ when n is odd and to $\mathbf{Z}/k\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ when n is even.

Proof. It is enough to do the calculation for $H_{(\infty)}^+$ the pullback of $\mathrm{GL}^+(n, \mathbf{R})$ in the universal cover of $\mathrm{Sp}(2n, \mathbf{R})$. The identity component of $H_{(\infty)}^+$ is then the cover of $\mathrm{GL}^+(n, \mathbf{R})$ corresponding to the subgroup of $\pi_1(\mathrm{GL}^+(n, \mathbf{R}))$ which is the kernel of the map $\pi_1(\mathrm{GL}^+(n, \mathbf{R})) \rightarrow \pi_1(\mathrm{Sp}(2n, \mathbf{R}))$. Since this map is zero the identity component of $H_{(\infty)}^+$ is isomorphic to $\mathrm{GL}^+(n, \mathbf{R})$ and the cover $H_{(\infty)}^+ \rightarrow \mathrm{GL}^+(n, \mathbf{R})$ is the trivial cover. Hence $\pi_0(H_{(\infty)}^+)$ is isomorphic to $Z \cong \pi_1(\mathrm{Sp}(2n, \mathbf{R}))$ the kernel of $\widetilde{\mathrm{Sp}}(2n, \mathbf{R}) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$.

Clearly $\pi_0(H_{(k)})/\pi_0(H_{(k)}^+) \cong \pi_0(\mathrm{GL}(n, \mathbf{R}))$ so $\pi_0(H_{(k)}^+)$ is the subgroup of index 2 in $\pi_0(H_{(k)})$. Determining the latter group is then easy. \square

The characteristic classes for a $(\mathrm{Sp}(2n, \mathbf{R})_{(k)}, H_{(k)})$ -Anosov representation ρ are then the obstruction classes $o_1(\rho)$ in $H^1(T^1\Sigma, \pi_0(H_{(k)}))$ and $o_2(\rho)$ in $H^2(T^1\Sigma, \pi_1(H_{(k)0})) \cong H^2(T^1\Sigma, \mathbf{F}_2)$. They project to the first and second Stiefel-Whitney classes $sw_1(\pi \circ \rho)$ and $sw_2(\pi \circ \rho)$ (in fact $o_2(\rho) = sw_2(\pi \circ \rho)$). A pair (c_1, c_2) in $H^1(T^1\Sigma, \pi_0(H_{(k)})) \times H^2(T^1\Sigma, \pi_1(H_{(k)0}))$ is called *n-admissible* if it projects to a *n-admissible* pair in $H^1(T^1\Sigma, \mathbf{F}_2) \times H^2(T^1\Sigma, \mathbf{F}_2)$ (Definition 4.19).

Theorem 4.23. *The space $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})_{(k)})$ is nonempty if and only if k divides $n(g(\Sigma) - 1)$. In that case the map*

$$\begin{aligned} (o_1, o_2) : \pi_0(\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})_{(k)}) \setminus \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})_{(k)})) \\ \longrightarrow H^1(T^1\Sigma; \pi_0(H_{(k)})) \times H^2(T^1\Sigma; \pi_1(H_{(k)0})) \end{aligned}$$

is the bijection onto the space of n-admissible pairs. The number of components of maximal representations is then:

$$\#\pi_0(\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})_{(k)})) = \begin{cases} (3 \times 2^{2g} + 2g(\Sigma) - 4)k^{2g} & \text{if } n = 2, \\ 3(2k)^{2g} & \text{if } n > 2. \end{cases}$$

Proof. The obstruction to lifting a representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ to the cover $\mathrm{Sp}(2n, \mathbf{R})_{(k)} \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ is the image of the Toledo number $\tau(\rho)$ by the map $\mathbf{Z} \cong H^2(\Sigma; \pi_1(\mathrm{Sp}(2n, \mathbf{R}))) \rightarrow H^2(\Sigma; Z_k) \cong \mathbf{Z}/k\mathbf{Z}$. Hence ρ lifts if and only if $\tau(\rho)$ is in $k\mathbf{Z}$. Since maximal representations have Toledo number $n(g - 1)$, this gives the restriction on k .

Lemma 4.22 and Lemma 4.9 imply that the map $H^1(\Sigma; Z_k) \rightarrow H^1(T^1\Sigma; \pi_0(H_{(k)}))$ is injective. The result follows then from Proposition 4.8. \square

It is sometimes possible to calculate the number of components of maximal representations in $\mathrm{PSp}(2n, \mathbf{R})$. Note that

$$\pi_1(\mathrm{PSp}(2n, \mathbf{R})) = \begin{cases} \mathbf{Z} & \text{--if } n \text{ is odd,} \\ \mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \text{--if } n \text{ is even,} \end{cases}$$

and contains $\pi_1(\mathrm{Sp}(2n, \mathbf{R}))$ as a subgroup of index 2. Moreover the bound for the Toledo number $\tau(\rho)$ of a representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{PSp}(2n, \mathbf{R})$ are

$$\begin{aligned} |\tau(\rho)| &\leq 2n(g-1) & \text{--if } n \text{ is odd,} \\ |\tau(\rho)| &\leq n(g-1) & \text{--if } n \text{ is even.} \end{aligned}$$

Hence maximal representations $\pi_1(\Sigma) \rightarrow \mathrm{PSp}(2n, \mathbf{R})$ always lift when n is odd. In that case $\mathrm{PGL}(n, \mathbf{R})$ is connected and its fundamental group is

$$\pi_1(\mathrm{PGL}(n, \mathbf{R})) = \mathbf{Z}/2\mathbf{Z}.$$

Therefore, for a maximal representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{PSp}(2n, \mathbf{R})$ (with n odd), only the second obstruction class $o_2(\rho)$ in $H^2(T^1\Sigma; \mathbf{F}_2)$ is relevant. Since ρ lifts to $\rho' : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$, $o_2(\rho) = sw_2(\rho')$ satisfies the conditions of Proposition 4.18: $o_2(\rho) \in H^2(\Sigma; \mathbf{F}_2)$.

Theorem 4.24. *Let n be an odd integer, $n \geq 3$.*

Then the space $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{PSp}(2n, \mathbf{R}))$ has 3 connected components. Furthermore $o_2 : \pi_0(\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{PSp}(2n, \mathbf{R})) \setminus \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{PSp}(2n, \mathbf{R}))) \rightarrow H^2(\Sigma; \mathbf{F}_2)$ is a bijection.

The proof is straightforward since Proposition 4.8 applies.

For the case when n is even, one gets

Proposition 4.25. *Let n be an even integer, $n \geq 2$.*

Then the image of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ in $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSp}(2n, \mathbf{R}))$ is the union of

- (i) $1 + (2g-2) + (2^{2g}-1)$ connected components of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{PSp}(2n, \mathbf{R}))$ when $n = 2$,
- (ii) $1 + 2 + (2^{2g} - 1)$ components of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{PSp}(2n, \mathbf{R}))$ when $n > 2$.

The above theorem and this proposition imply Theorem 8 of the introduction.

Proof. All the components of $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ projects to the same component in $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{PSp}(2n, \mathbf{R}))$. Let \mathcal{C} be a connected component of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) \setminus \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$. We need to understand how many components are identified with \mathcal{C} under the projection p , i.e. the number of components of $\mathcal{D} = p^{-1}(p(\mathcal{C}))$. Note that, denoting $Z = \{\pm \mathrm{Id}\}$ the center of $\mathrm{Sp}(2n, \mathbf{R})$,

$$\mathcal{D} = \{\eta \cdot \rho \mid \rho \in \mathcal{C}, \eta \in \mathrm{Hom}(\pi_1(\Sigma), Z)\}.$$

Let sw_1 in $H^1(T^1\Sigma; \mathbf{F}_2)$ the first Stiefel-Whitney class of any representation in \mathcal{C} . We shall prove:

$$\begin{aligned} |\pi_0(\mathcal{D})| &= 2 & \text{--if } sw_1 \neq 0 \\ |\pi_0(\mathcal{D})| &= 1 & \text{--if } sw_1 = 0. \end{aligned}$$

By Theorems 6 and 7 it is enough to show that, for any $\rho \in \mathcal{C}$ and $\eta \in \mathrm{Hom}(\pi_1(\Sigma), Z)$,

- $sw_1(\eta \cdot \rho) = sw_1(\rho)$.
- if $sw_1(\rho) = 0$, $sw_2(\eta \cdot \rho) = sw_2(\rho)$ (or $e_\gamma(\eta \cdot \rho) = e_\gamma(\rho)$ when $n = 2$).

– if $sw_1(\rho) \neq 0$, there exists η' such that $sw_2(\eta' \cdot \rho) \neq sw_2(\rho)$.

If D_η is the flat line bundle associated with η , the Lagrangian reduction for $\eta \cdot \rho$ is

$$L(\eta \cdot \rho) = D_\eta \otimes L(\rho).$$

From Proposition B.6 the Stiefel-Whitney classes are

$$sw_1(\eta \cdot \rho) = sw_1(\rho) + nsw_1(D_\eta) = sw_1(\rho)$$

since n is even, and

$$sw_2(\eta \cdot \rho) = sw_2(\rho) + (n-1)sw_1(D_\eta) \smile sw_1(\rho) + \frac{n(n-1)}{2}sw_1(D_\eta) \smile sw_1(D_\eta).$$

In the above equality every class is in the cohomology of Σ . Since the map $H^1(\Sigma; \mathbf{F}_2) \otimes H^1(\Sigma; \mathbf{F}_2) \rightarrow H^2(\Sigma; \mathbf{F}_2) \cong \mathbf{F}_2$, $a \otimes b \mapsto a \smile b$ is a nondegenerate antisymmetric form on $H^1(\Sigma; \mathbf{F}_2)$, one has $sw_1(D_\eta) \smile sw_1(D_\eta) = 0$ and thus

$$sw_2(\eta \cdot \rho) = sw_2(\rho) + sw_1(D_\eta) \smile sw_1(\rho).$$

When $sw_1(\rho) = 0$ this equality become $sw_2(\eta \cdot \rho) = sw_2(\rho)$. When $sw_1(\rho) \neq 0$, by nondegeneracy, there exists η' with $sw_1(D_{\eta'}) \smile sw_1(\rho) \neq 0$. Therefore $sw_2(\eta' \cdot \rho) \neq sw_2(\rho)$. When $n = 2$ and $sw_1(\rho)$ is zero, the Euler classes $e_\gamma(\eta \cdot \rho)$ and $e_\gamma(\rho)$ are the Euler classes of the S^1 -bundles $S_{\eta \cdot \rho}$ and S_ρ . The above formula for $L(\eta \cdot \rho)$ implies that $S_{\eta \cdot \rho} = S_\eta \times_{S^1} S_\rho$ where S_η is the flat S^1 -bundle associated with η . Therefore one has $e_\gamma(\eta \cdot \rho) = e(S_\eta) + e_\gamma(\rho)$. Since $e(S_\eta)$ depends continuously on η in $\text{Hom}(\pi_1(\Sigma), S^1) \cong (S^1)^{2g}$, this Euler class is zero. \square

5. COMPUTATIONS OF THE INVARIANTS

In this section we calculate the invariants for the maximal representations introduced in Section 3. First the computation is performed for standard maximal representations (Sections 5.1, 5.2 and 5.3). In Section 5.4 we deduce Theorems 6 and 11 and Corollary 13 of the introduction. In Section 5.5 we calculate the invariants for the hybrid representations defined in Section 3.3.1.

5.1. Irreducible Fuchsian representations. Let $\iota : \pi_1(\Sigma) \rightarrow \text{SL}(2, \mathbf{R})$ be a discrete embedding and $\rho_{irr} = \phi_{irr} \circ \iota : \pi_1(\Sigma) \rightarrow \text{Sp}(2n, \mathbf{R})$ an irreducible Fuchsian representation. We observed in Fact 3.2.(i) that the Lagrangian reduction $L^s(\rho_{irr})$ is given by

$$L^s(\rho_{irr}) = L^s(\iota)^{2n-1} \oplus L^s(\iota)^{2n-3} \oplus \dots \oplus L^s(\iota),$$

where $L^s(\iota)$ is the line bundle associated with ι .

Therefore, by the multiplicative properties of Stiefel-Whitney classes, the first and second Stiefel-Whitney classes are given by

$$sw_1(\rho_{irr}) = sw_1(L^s(\rho_{irr})) = \left(\sum_{i=1}^n (2i-1) \right) sw_1(L^s(\iota)) = nsw_1(L^s(\iota)),$$

and

$$\begin{aligned} sw_2(\rho_{irr}) &= sw_2(L^s(\rho_{irr})) = \frac{n(n-1)}{2} sw_1(L^s(\iota)) \smile sw_1(L^s(\iota)) \\ &= \frac{n(n-1)}{2} (g-1) \pmod{2}, \end{aligned}$$

where the last equality follows from the calculation for $n = 2$; in that case $sw_1(L^s(\iota)) \smile sw_1(L^s(\iota)) = sw_2(\rho_{irr}) = e_\gamma(\rho_{irr}) \pmod{2} = (g-1) \pmod{2}$ by Proposition 5.10. Note that in particular $sw_1(\rho_{irr}) = 0$ if n is even.

5.2. Diagonal Fuchsian representations. Let $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ be a discrete embedding and $\rho_\Delta = \phi_\Delta \circ \iota : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ a diagonal Fuchsian representation. We observed in Facts 3.3.(i) that the Lagrangian reduction $L^s(\rho_\Delta)$ is given by

$$L^s(\rho_\Delta) = L^s(\iota) \oplus \cdots \oplus L^s(\iota),$$

where $L^s(\iota)$ is the line bundle associated with ι .

Therefore the first and second Stiefel-Whitney classes are given by

$$sw_1(\rho_\Delta) = sw_1(L^s(\rho_\Delta)) = nsw_1(L^s(\iota)),$$

and

$$\begin{aligned} sw_2(\rho_\Delta) = sw_2(L^s(\rho_\Delta)) &= \frac{n(n-1)}{2} sw_1(L^s(\iota)) \smile sw_1(L^s(\iota)) \\ &= \frac{n(n-1)}{2} (g-1) \pmod{2}. \end{aligned}$$

Again, $sw_1(\rho_\Delta) = 0$ if n is even.

5.3. Twisted diagonal representations. The aim of this section is to prove the

Proposition 5.1. *Let $n \geq 2$. Let $(\alpha, \beta) \in H^1(T^1\Sigma; \mathbf{F}_2) \times H^2(T^1\Sigma; \mathbf{F}_2)$ be an n -admissible pair (see Definition 4.19). If $n = 2$ and $\alpha = 0$ we suppose furthermore that $\beta = (g-1) \pmod{2}$.*

Then there exists a twisted diagonal representation

$$\rho_\Theta : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$$

such that $sw_1(\rho_\Theta) = \alpha$ and $sw_2(\rho_\Theta) = \beta$.

Remark 5.2. Note that when $n = 2$ twisted diagonal representations ρ_Θ with $sw_1(\rho_\Theta) = 0$ will always have Euler class $e_{\gamma, L_{0+}}(\rho_\Theta) = (g-1)[\Sigma] \in H^2(T^1\Sigma; \mathbf{Z})$ (Proposition 5.10) so that the second Stiefel-Whitney class is $sw_2(\rho_\Theta) = g-1 \pmod{2}$.

Let $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ be a discrete embedding, $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{O}(n)$ an orthogonal representation and $\rho_\Theta = \iota \otimes \Theta : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ the corresponding twisted diagonal representation. By Fact 3.5.(ii) the Lagrangian reduction $L^s(\rho_\Theta)$ is given by

$$L^s(\rho_\Theta) = L^s(\iota) \otimes \overline{W}_\Theta,$$

where $L^s(\iota)$ is the line bundle associated with ι and \overline{W}_Θ is the pullback to $T^1\Sigma$ of the flat n -plane bundle W_Θ over Σ associated with Θ .

We have $sw_i(\overline{W}_\Theta) = \pi^* sw_i(W_\Theta)$, where $\pi^* : H^i(\Sigma; \mathbf{F}_2) \rightarrow H^i(T^1\Sigma; \mathbf{F}_2)$ is induced by the projection $\pi : T^1\Sigma \rightarrow \Sigma$.

Thus, to compute the invariants for twisted diagonal representations we will need to study the first and second Stiefel-Whitney classes of orthogonal representations $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{O}(n)$. For such a Θ denote by $sw_i(\Theta) = sw_i(W_\Theta) \in H^i(\Sigma; \mathbf{F}_2)$ the Stiefel-Whitney classes. We first note

Lemma 5.3. *Let $m \leq n$ and let $\phi : \mathrm{O}(m) \rightarrow \mathrm{O}(n)$ be the injection as the subgroup fixing pointwise a $(n-m)$ -dimensional subspace. Then for any $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{O}(m)$ and any i one has*

$$sw_i(\Theta) = sw_i(\phi \circ \Theta).$$

5.3.1. *The first Stiefel-Whitney class of an orthogonal representation.* Let $\det : \mathrm{O}(n) \rightarrow \{\pm 1\}$ be the determinant homomorphism. Then the homomorphism

$$\det \circ \Theta : \pi_1(\Sigma) \longrightarrow \{\pm 1\}$$

corresponds to the first Stiefel-Whitney class $sw_1(\Theta) \in H^1(\Sigma; \mathbf{F}_2)$ under the identification $H^1(\Sigma; \mathbf{F}_2) \cong \mathrm{Hom}(H^1(\Sigma; \mathbf{Z}), \mathbf{F}_2) \cong \mathrm{Hom}(\pi_1(\Sigma), \{\pm 1\})$. In particular, the first Stiefel-Whitney class is zero if the representation has image in $\mathrm{SO}(n)$.

Lemma 5.4. *Let $\alpha \in H^1(\Sigma; \mathbf{F}_2)$. Then, for any $n \geq 1$, there exists a representation $\Theta_\alpha : \pi_1(\Sigma) \rightarrow \mathrm{O}(n)$ such that $sw_1(\Theta_\alpha) = \alpha$.*

Proof. The result is obvious for $n = 1$ since \det is then an isomorphism. For general n apply Lemma 5.3. \square

5.3.2. *The second Stiefel-Whitney class of an orthogonal representation.* Let $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{O}(n)$ be an orthogonal representation. The second Stiefel-Whitney class of Θ can be described as the obstruction to lifting the representation ρ to the nontrivial double cover $\mathrm{Pin}(n)$:

$$1 \rightarrow \{\pm 1\} \rightarrow \mathrm{Pin}(n) \rightarrow \mathrm{O}(n) \rightarrow 1.$$

This obstruction class lies in $H^2(\Sigma, \{\pm 1\}) = \{\pm 1\} \cong \mathbf{F}_2$ and can be explicitly calculated via the following procedure. First, fix a standard presentation

$$\pi_1(\Sigma) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \rangle.$$

Second, since the extension $\mathrm{Pin}(n) \rightarrow \mathrm{O}(n)$ is central, the commutator map for $\mathrm{Pin}(n)$ factors through $\mathrm{O}(n)$:

$$[\cdot, \cdot]^\sim : \mathrm{O}(n) \times \mathrm{O}(n) \longrightarrow \mathrm{Pin}(n).$$

The second Stiefel-Whitney class $sw_2(\Theta) \in H^2(\Sigma; \mathbf{F}_2)$ is given by

$$\prod_{i=1}^g [\Theta(a_i), \Theta(b_i)]^\sim \in \mathbf{F}_2.$$

Lemma 5.5. *Let $\alpha \in H^1(\Sigma; \mathbf{F}_2)$ and $\beta \in H^2(\Sigma; \mathbf{F}_2)$.*

– *If $\alpha \neq 0$ or $\beta = 0$ then, for any $n \geq 2$, there exists a representation $\Theta_{\alpha, \beta} : \pi_1(\Sigma) \rightarrow \mathrm{O}(n)$ such that $sw_1(\Theta_{\alpha, \beta}) = \alpha$ and $sw_2(\Theta_{\alpha, \beta}) = \beta$.*

– *If $\alpha = 0$ and $\beta \neq 0$ then, for any $n \geq 3$, there exists a representation $\Theta_{\alpha, \beta}$ such that $sw_1(\Theta_{\alpha, \beta}) = \alpha$ and $sw_2(\Theta_{\alpha, \beta}) = \beta$.*

Furthermore, in every case, $\Theta_{\alpha, \beta}$ can be chosen to have finite image in $\mathrm{O}(n)$.

Similar statements may be found in [41, Sec. 4.2].

Remark 5.6. Note that for Θ in $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{O}(2))$ the equality $sw_1(\Theta) = 0$ automatically implies $sw_2(\Theta) = 0$.

Proof. Since the mapping class group acts transitively on $H^1(\Sigma; \mathbf{F}_2) \setminus \{0\}$ and trivially on $H^2(\Sigma; \mathbf{F}_2)$ and in view of Lemma 5.3, we need to construct the following representations:

- (i) $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{O}(1)$ with $sw_1(\Theta) = 0$ and $sw_2(\Theta) = 0$.
- (ii) $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{O}(1)$ with $sw_1(\Theta) \neq 0$ and $sw_2(\Theta) = 0$.
- (iii) $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{O}(2)$ with $sw_1(\Theta) \neq 0$ and $sw_2(\Theta) \neq 0$.
- (iv) $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{O}(3)$ with $sw_1(\Theta) = 0$ and $sw_2(\Theta) \neq 0$.

By Lemma 5.4, the existence of representations required in (i) and (ii) is obvious.

For (iii) let us denote by $e^{i\theta}$ elements of $\mathrm{SO}(2)$ and by R an element of $\mathrm{O}(2) \setminus \mathrm{SO}(2)$. Then $\det(e^{i\theta}) = 1$ and $\det(Re^{i\theta}) = -1$.

We define the representation $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{O}(2)$ by

$$\Theta(a_1) = R, \quad \Theta(b_1) = e^{i\pi}, \quad \text{and} \quad \Theta(a_i) = \Theta(b_i) = 1 \quad \text{for any } i > 1.$$

The relation $\Pi_{i=1}^g[\Theta(a_i), \Theta(b_i)] = 1$ is obviously satisfied since $e^{i\pi}$ is central in $\mathrm{O}(2)$. Clearly $sw_1(\Theta) \neq 0$.

The group $\mathrm{Pin}(2)$ is in this case isomorphic to $\mathrm{O}(2)$ and the restriction to $\mathrm{SO}(2)$ of the cover $\mathrm{O}(2) \cong \mathrm{Pin}(2) \rightarrow \mathrm{O}(2)$ is $\mathrm{SO}(2) \rightarrow \mathrm{SO}(2)$, $x \mapsto x^2$, the image of R being R . We can hence lift $\Theta(b_1)$ to $e^{i\pi/2}$ and $\Theta(a_1)$ to R . Then we have

$$sw_2(\Theta) = \Pi_{i=1}^g[\Theta(a_i), \Theta(b_i)]^\sim = e^{-i\pi} = -1.$$

For (iv) we are looking for a representation $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{SO}(3)$ (i.e. $sw_1(\Theta) = 0$) which does not lift to $\mathrm{Spin}(3)$. Let us realize $\mathrm{Spin}(3) \cong S^3$ as the quaternions of norm one. The covering map $\mathrm{Spin}(3) \rightarrow \mathrm{SO}(3)$ is realized by the action by conjugation on the space of imaginary quaternions.

Let us denote by $\{1, i, j, k\}$ the standard basis of the quaternions. We define a representation $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{SO}(3)$ by sending a_1 to the projection in $\mathrm{SO}(3)$ of i , b_1 to the projection of j and all other generators of $\pi_1(\Sigma)$ to the trivial element. Since $[\Theta(a_1), \Theta(b_1)]^\sim = [i, j] = -1$ this defines a homomorphism into $\mathrm{SO}(3)$ which does not lift to $\mathrm{Spin}(3)$. \square

5.3.3. Proof of Proposition 5.1. We have to show that we can realize all 2^{2g} different choices for $sw_1(\rho)$ in the fixed coset of $H^1(T^1\Sigma; \mathbf{F}_2)$ and the 2 choices for $sw_2(\rho)$ in the image of $H^2(\Sigma; \mathbf{F}_2)$ in $H^2(T^1\Sigma; \mathbf{F}_2)$.

Let $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ be the fixed discrete embedding, $\Theta : \pi_1(\Sigma) \rightarrow \mathrm{O}(n)$ an orthogonal representation and $\rho_\Theta : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ the corresponding twisted diagonal representation.

The following formulas hold for the first and second Stiefel-Whitney classes [43, Corollary 5.4] (Proposition B.6 provides a short proof for the reader's convenience):

$$(5.7) \quad sw_1(\rho_\Theta) = sw_1(L^s(\iota) \otimes \overline{W}) = nsw_1(L^s(\iota)) + sw_1(\overline{W})$$

$$sw_2(\rho_\Theta) = sw_2(L^s(\iota) \otimes \overline{W}) = \frac{n(n-1)}{2}sw_1(L^s(\iota)) \smile sw_1(L^s(\iota)) + (n-1)sw_1(L^s(\iota)) \smile sw_1(\overline{W}) + sw_2(\overline{W}),$$

where \overline{W} is the pullback to $T^1\Sigma$ of the flat n -plane bundle over Σ associated with Θ . Note that $sw_i(\overline{W})$ is the image of $sw_i(\Theta)$ under the natural map in cohomology.

Lemma 5.5 implies that by choosing different $\mathrm{O}(n)$ -representations for Θ we can realize $2^{2g} \times 2$ different choices for $sw_1(\Theta)$ and $sw_2(\Theta)$. Since $nsw_1(\iota)$ is fixed, as we vary $sw_1(\overline{W})$ over the 2^{2g} distinct classes in the image of $H^1(\Sigma; \mathbf{F}_2)$ in $H^1(T^1\Sigma; \mathbf{F}_2)$ we realize all possible 2^{2g} classes in the $H^1(\Sigma; \mathbf{F}_2)$ -coset in $H^1(T^1\Sigma; \mathbf{F}_2)$ determined by $nsw_1(\iota)$. Similarly for any $sw_1(\overline{W})$ we can realize the two possibilities for $sw_2(\overline{W})$, so we can realize the two possibilities for sw_2 by an appropriately chosen twisted diagonal maximal representation $\rho_\Theta : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$.

5.4. Consequences for maximal representations. Let us now derive some of the results stated in the introduction.

Every representation in the Hitchin component is purely loxodromic [36, Prop. 3.4], therefore twisted diagonal representations are never contained in Hitchin components. Proposition 5.1 implies then that for $n \geq 3$ the space

$$\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) \setminus \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$$

consists of at least 2×2^{2g} connected components. As the total number of connected components of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$, if $n \geq 3$, is 3×2^{2g} [18, Theorem 8.7] and as $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ has 2^{2g} components, we conclude that the space $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) \setminus \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ has 2×2^{2g} components and that the first and second Stiefel-Whitney classes of the Anosov bundle associated with the representation distinguish them; this proves Theorem 6. In the 2^{2g} Hitchin components any representation can be deformed into an irreducible Fuchsian representation. The remaining 2×2^{2g} connected components all contain a twisted diagonal representation. This gives Theorem 11 and Corollary 12 of the introduction. Corollary 13 then follows directly from the computations in Section 5.1 and Section 5.2.

5.5. Hybrid representations. The goal of this section is to prove the following

Theorem 5.8. *Let $\gamma \in \pi_1(\Sigma) \setminus \{1\}$ and $L_{0+} \in \mathcal{L}_+$ an oriented Lagrangian subspace of \mathbf{R}^4 . Then the relative Euler class distinguishes the connected components of*

$$\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \setminus \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})).$$

More precisely, given any l in $\mathbf{Z}/(2g-2)\mathbf{Z} \cong H^2(T^1\Sigma; \mathbf{Z})^{tor}$,

- if $l \neq (g-1)$ the set $e_{\gamma, L_{0+}}^{-1}(\{l\})$ is a connected component of the space $\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$. Every representation in $e_{\gamma, L_{0+}}^{-1}(\{l\})$ can be deformed to a k -hybrid representation, where $k = g-1-l \pmod{2g-2}$;
- the set $e_{\gamma, L_{0+}}^{-1}(\{g-1\})$ is the union of the connected component of $\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ containing diagonal Fuchsian representations in $\mathrm{Sp}(4, \mathbf{R})$ and of the Hitchin components $\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$.

For simplicity we restrict the discussion to the case of k -hybrid representations which are constructed from the decomposition of the surface $\Sigma = \Sigma_l \cup \Sigma_r$ along a simple closed separating geodesic γ (see Section 3.3.1). The general computations follow by a straightforward extension of this case.

The first step is to calculate the first Stiefel-Whitney class.

Proposition 5.9. *Let $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ be a k -hybrid representation as defined in Section 3.3.1, then*

$$sw_1(\rho) = 0.$$

Proof. By Proposition 4.18 we have $sw_1(\rho) \in H^1(\Sigma; \mathbf{F}_2) \subset H^1(T^1\Sigma; \mathbf{F}_2)$, thus it is sufficient to show that $sw_1(\rho)$ is zero on a basis of the first homology group of Σ .

Let $\pi_1(\Sigma) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \rangle$ be a standard presentation, then $a_1, b_1, \dots, a_g, b_g$ form a basis of $H^1(\Sigma; \mathbf{F}_2)$. We can choose such a standard presentation of $\pi_1(\Sigma)$ with the property that for any element h of the generating set, either $h \in \pi_1(\Sigma_l)$ or $h \in \pi_1(\Sigma_r)$, where the hybrid representation ρ was constructed with respect to a decomposition $\Sigma = \Sigma_l \cup \Sigma_r$. Then the sign of $\det(\rho(h)|_{L^s(h)})$, where $L^s(h)$ is the attracting Lagrangian for $\rho(h)$, is positive for every element h in the

above generating set. This can be checked independently for the irreducible Fuchsian representation ρ_{irr} and (deformations of) the diagonal Fuchsian representation ρ_Δ . In view of Lemma 4.11 this implies $sw_1(\rho) = 0$. \square

Proposition 5.10. *Let $\Sigma = \Sigma_l \cup \Sigma_r$ and $\rho = \rho_l * \rho_r : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ be a hybrid representation as defined in Section 3.3.1.*

Let $\gamma \in \pi_1(\partial\Sigma_l)$ and let $(\epsilon_i)_{i=1,\dots,4}$ one of the basis appearing in Definition 3.10. Suppose that $L_{0+} = \langle \epsilon_1, \epsilon_2 \rangle$ is an attracting fixed point for $\rho(\gamma)$.

Then

$$e_{\gamma, L_{0+}}(\rho) = (g - 1 - \chi(\Sigma_l))[\Sigma] \in H^2(T^1\Sigma; \mathbf{Z}).$$

This proposition implies easily Theorem 5.8.

Remark 5.11. The result is valid also when Σ_l of Σ_r is empty.

Remark 5.12. If the pair (ρ_l, ρ_r) is negatively adjusted with respect to γ (Definition 3.10), then the Euler number is

$$e_{\gamma, L_{0+}}(\rho) = (g - 1 + \chi(\Sigma_l))[\Sigma] \in H^2(T^1\Sigma; \mathbf{Z}).$$

5.5.1. *Reduction to the group $\widehat{\pi_1(\Sigma)}$.* Our strategy to prove Proposition 5.10 is to change the representation ρ slightly.

We denote by $\widehat{\pi_1(\Sigma)}$ the group $\{\pm 1\} \times \pi_1(\Sigma)$; using the morphism $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ there is an embedding $\widehat{\iota} : \widehat{\pi_1(\Sigma)} \rightarrow \mathrm{SL}(2, \mathbf{R})$, with $\widehat{\iota}(-1) = -\mathrm{Id}_2$ and $\widehat{\iota}|_{\pi_1(\Sigma)} = \iota$. Observe that

$$\widehat{\iota}(\widehat{\pi_1(\Sigma)}) \backslash \mathrm{SL}(2, \mathbf{R}) \cong \iota(\pi_1(\Sigma)) \backslash \mathrm{PSL}(2, \mathbf{R}) \cong T^1\Sigma,$$

so that the group $\widehat{\pi_1(\Sigma)}$ is a quotient of the group $\pi_1(T^1\Sigma)$ and hence the notion of Anosov representations and their invariants (see Section 2 and Section 4) are well defined for $\widehat{\pi_1(\Sigma)}$.

Let $\varepsilon : \widehat{\pi_1(\Sigma)} \rightarrow \{\pm 1\}$ be the projection onto the first factor; for any representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ we define a representation $\varepsilon \otimes \rho$ of $\widehat{\pi_1(\Sigma)}$ into $\mathrm{Sp}(4, \mathbf{R})$ by setting $\varepsilon \otimes \rho(x, \gamma) = \varepsilon(x)\rho(\gamma)$. The relations between the invariants of ρ and of $\varepsilon \otimes \rho$ are discussed in Appendix A.3. In view of these relations Proposition 5.10 follows from:

Proposition 5.13. *Let $\rho, \Sigma_l, \Sigma_r, \gamma$ and L_{0+} be as in Proposition 5.10.*

Then

$$e_{\gamma, L_{0+}}(\varepsilon \otimes \rho) = -\chi(\Sigma_l)[\Sigma].$$

The rest of this section is devoted to the proof of this proposition.

5.5.2. *Constructing sections.* In order to calculate the Euler class for $\varepsilon \otimes \rho$ we will construct a lift of $e_{\gamma, L_{0+}}(\varepsilon \otimes \rho)$ under the connecting homomorphism $H^1(T^1\Sigma|_\gamma; \mathbf{Z}) \rightarrow H^2(T^1\Sigma; \mathbf{Z})$ appearing in the Mayer-Vietoris sequence (Appendix B.1) and then calculate this lift in $H^1(T^1\Sigma|_\gamma; \mathbf{Z}) \cong \mathbf{Z}^2$. Such a lift can be constructed from trivializations of the Lagrangian bundle.

Proposition 5.14. *Let $\rho, \Sigma_l, \Sigma_r, \gamma$ and L_{0+} be as in Proposition 5.10. Let $L_+^s(\varepsilon \otimes \rho)$ be the oriented Lagrangian reduction for the flat $\varepsilon \otimes \rho$ -flat bundle over $T^1\Sigma$.*

Then the restrictions of $L_+^s(\varepsilon \otimes \rho)$ to both $T^1\Sigma|_{\Sigma_l}$ and $T^1\Sigma|_{\Sigma_r}$ are trivial.

This proposition is a consequence of the following three lemmas.

Lemma 5.15. *Let $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ be a maximal representation and let $\phi : \mathrm{SL}(2, \mathbf{R}) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ be a homomorphism such that $\rho = \phi \circ \iota$ is maximal.*

Then the oriented Lagrangian bundle $L_+^s(\varepsilon \otimes \rho)$ is trivial.

Proof. First we observe that $\varepsilon \otimes \rho = \phi \circ \hat{\iota}$ with $\hat{\iota} = \varepsilon \otimes \iota$. In this situation the map defining the oriented Lagrangian bundle is the equivariant map

$$\begin{aligned} \mathrm{SL}(2, \mathbf{R}) &\longrightarrow \mathcal{L}_+ \\ g &\longmapsto \phi(g) \cdot L_{0+}, \end{aligned}$$

where L_{0+} is the attracting Lagrangian fixed by $\rho(\gamma)$.

An equivariant map $\mathrm{SL}(2, \mathbf{R}) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ that trivializes the corresponding bundle is given simply by

$$g \longmapsto \phi(g)^{-1}. \quad \square$$

Remark 5.16. (i) A *trivialization* of a symplectic bundle $E = \pi_1(M) \backslash (\widetilde{M} \times \mathbf{R}^4)$ is an isomorphism with the trivial bundle $M \times \mathbf{R}^4$. At the level of the universal cover this is the same as an isomorphism from $\widetilde{M} \times \mathbf{R}^4$ to $\widetilde{M} \times \mathbf{R}^4$ intertwining the action of $\pi_1(M)$ by the representation $\rho : \pi_1(M) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ and the trivial action. This is given by a map $\varphi : \widetilde{M} \rightarrow \mathrm{Sp}(4, \mathbf{R})$ satisfying the equivariance equation: $\varphi(\gamma \cdot \tilde{m}) = \varphi(\tilde{m})\rho(\gamma)^{-1}$ for all γ in $\pi_1(M)$ and \tilde{m} in \widetilde{M} .

(ii) Let L be a subbundle E or, what amounts to the same, an invariant subbundle of $\widetilde{M} \times \mathbf{R}^4$, i.e. a ρ -equivariant map $\zeta : \widetilde{M} \rightarrow \mathrm{Gr}_2(\mathbf{R}^4)$. The trivialization φ induces furthermore a trivialization of L if the map $\tilde{m} \mapsto \varphi(\tilde{m}) \cdot \zeta(\tilde{m})$ is constant.

Lemma 5.17. *Let ρ_0 and ρ_1 be two homotopic maximal representations. Then the Lagrangian bundles $L^s(\rho_0)$ and $L^s(\rho_1)$ are homotopic and hence they are isomorphic.*

If the first Stiefel-Whitney class $sw_1(\rho_0) = sw_1(\rho_1)$ is zero, then the corresponding oriented Lagrangian bundles are also isomorphic.

Proof. This is a consequence of the fact that the equivariant positive curve depends continuously on the representation (Fact 2.26) and the fact that homotopic bundles are isomorphic [42, p. 53]. \square

Lemma 5.18. *Let ρ and ρ' be two maximal representations with zero first Stiefel-Whitney class and such that $\rho|_{\pi_1 \Sigma'} = \rho'|_{\pi_1 \Sigma'}$ for a subsurface $\Sigma' \subset \Sigma$.*

Then the two bundles $L_+^s(\varepsilon \otimes \rho)$ and $L_+^s(\varepsilon \otimes \rho')$ are homotopic when restricted to $T^1 \Sigma|_{\Sigma'}$.

Proof. Note that if $L^s(\varepsilon \otimes \rho)$ is homotopic to $L^s(\varepsilon \otimes \rho')$ then $L_+^s(\varepsilon \otimes \rho)$ is homotopic to $L_+^s(\varepsilon \otimes \rho')$. Therefore it is sufficient to prove the result without considering orientations. Let D_ε be the flat line bundle associated with ε , so that $L^s(\varepsilon \otimes \rho) = D_\varepsilon \otimes L^s(\rho)$ and $L^s(\varepsilon \otimes \rho') = D_\varepsilon \otimes L^s(\rho')$. Thus, we need to show that $L^s(\rho)|_{M'}$ is homotopic to $L^s(\rho')|_{M'}$ with $M' = T^1 \Sigma|_{\Sigma'} \subset M = T^1 \Sigma$.

Let $\widetilde{\Sigma}'$ be the universal cover of Σ' ; it can be realized as a $\pi_1(\Sigma')$ -invariant subset of $\widetilde{\Sigma}$. More precisely, under an identification $\widetilde{\Sigma} \cong \mathbb{H}_2$ we can set $\widetilde{\Sigma}' = \mathrm{Conv}(\Lambda_{\pi_1(\Sigma')})$ the convex hull of the limit set $\Lambda_{\pi_1(\Sigma')}$ of $\pi_1(\Sigma')$ in the boundary $\partial \mathbb{H}_2$ of the hyperbolic plane.

The manifold $\overline{M} = T^1 \widetilde{\Sigma}$ is a $\pi_1(\Sigma)$ -cover of M and we set $\overline{M}' = T^1 \widetilde{\Sigma}|_{\widetilde{\Sigma}'} \subset T^1 \widetilde{\Sigma} = \overline{M}$ so that $M' \cong \pi_1(\Sigma') \backslash \overline{M}'$. When we identify the unit tangent bundle \overline{M} with

$\partial\pi_1(\Sigma)^{(3+)}$, the set of positively oriented triples of $\partial\pi_1(\Sigma)$, we can identify \overline{M}' with the subset of $\partial\pi_1(\Sigma)^{(3+)}$ whose projection to $\tilde{\Sigma}$ belongs to $\tilde{\Sigma}'$.

The bundle $L^s(\rho)$ is constructed via the ρ -equivariant map

$$\begin{aligned} \partial\pi_1(\Sigma)^{(3+)} &\xrightarrow{p} \partial\pi_1(\Sigma) \xrightarrow{\xi} \mathcal{L} \\ (t^s, t, t^u) &\longmapsto t^s \longmapsto \xi(t^s) \end{aligned}$$

where ξ is the positive ρ -equivariant curve. Similarly $L^s(\rho')$ is constructed from the positive ρ' -equivariant curve ξ' .

This means that the restriction $L^s(\rho)|_{M'}$ is constructed from the $\rho|_{\pi_1(\Sigma')}$ -equivariant map

$$\overline{M}' \xrightarrow{p|_{\overline{M}'}} \partial\pi_1(\Sigma) \xrightarrow{\xi} \mathcal{L}.$$

Conversely, any $\rho|_{\pi_1(\Sigma')}$ -equivariant continuous map $\partial\pi_1(\Sigma) \rightarrow \mathcal{L}$ defines a Lagrangian reduction of the symplectic \mathbf{R}^4 -bundle over M' . Hence we get a homotopy of the two bundles $L^s(\rho)|_{M'}$ and $L^s(\rho')|_{M'}$ once we have a $\rho|_{\pi_1(\Sigma')}$ -equivariant homotopy between the two maps

$$\begin{aligned} \overline{M}' &\xrightarrow{p|_{\overline{M}'}} \partial\pi_1(\Sigma) \xrightarrow{\xi} \mathcal{L} \\ \overline{M}' &\xrightarrow{p|_{\overline{M}'}} \partial\pi_1(\Sigma) \xrightarrow{\xi'} \mathcal{L}. \end{aligned}$$

For this it is sufficient to construct a $\rho|_{\pi_1(\Sigma')}$ -equivariant homotopy between the two positive maps ξ and ξ' . This is the content of the next lemma. \square

Lemma 5.19. *Let $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ be a maximal representation and let Σ' be a subsurface. Then the set*

$$\mathcal{C} = \{\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{L} \mid \xi \text{ is positive, continuous and } \rho|_{\pi_1(\Sigma')}\text{-equivariant}\}$$

is connected.

Proof. To simplify notation we assume that the boundary $\partial\Sigma' = \gamma$ consists of one component. Note first that the restriction to the limit set $\Lambda_{\pi_1(\Sigma')}$ of any curve ξ in \mathcal{C} is completely determined by the action of $\pi_1(\Sigma')$. The complement $\partial\pi_1(\Sigma) \setminus \Lambda_{\pi_1(\Sigma')}$ is a countable union of open intervals, which are transitively exchanged by the action of $\pi_1(\Sigma')$. The interval (t_γ^u, t_γ^s) , where t_γ^u and t_γ^s are the fixed points of γ in $\partial\pi_1(\Sigma)$, is one such interval. The map ξ is completely determined by its restriction to this interval.

Conversely given a positive $\rho|_{\langle\gamma\rangle}$ -equivariant continuous curve $\beta : (t_\gamma^u, t_\gamma^s) \rightarrow \mathcal{L}_+$, one obtains a continuous $\rho|_{\pi_1(\Sigma')}$ -equivariant positive curve $\partial\pi_1(\Sigma) \setminus \Lambda_{\pi_1(\Sigma')} \rightarrow \mathcal{L}$, which one shows to have a continuous extension $\partial\pi_1(\Sigma) \rightarrow \mathcal{L}$.

Thus the map:

$$\begin{aligned} \mathcal{C} &\longrightarrow \{\beta : (t_\gamma^u, t_\gamma^s) \rightarrow \mathcal{L} \mid \beta \text{ positive continuous and } \rho|_{\langle\gamma\rangle}\text{-equivariant}\} \\ \xi &\longmapsto \xi|_{(t_\gamma^u, t_\gamma^s)} \end{aligned}$$

is a homeomorphism. By Proposition A.3, this space is connected. \square

5.5.3. Calculating the Euler class. Proposition 5.14 gives precisely what is needed to construct a lift of the Euler class $e_{\gamma, L_{0+}}(\varepsilon \otimes \rho)$ under the connecting homomorphism $H^1(T^1\Sigma|_\gamma; \mathbf{Z}) \rightarrow H^2(T^1\Sigma; \mathbf{Z})$. We have (see Appendix B.1)

$$(5.20) \quad H^1(T^1\Sigma|_\gamma; \mathbf{Z}) \cong \mathrm{Hom}(\pi_1(T^1\Sigma|_\gamma), \mathbf{Z}) \cong \mathbf{Z} \times \mathbf{Z}.$$

The last identification sends $\phi \in \text{Hom}(\pi_1(T^1\Sigma|_\gamma), \mathbf{Z})$ to $(\phi(T_x^1\Sigma), \phi(\gamma))$, where the two circles $T_x^1\Sigma$ and γ are naturally considered as loops in $T^1\Sigma|_\gamma$.

Proposition 5.21. *Let ρ , Σ_l , Σ_r , γ and L_{0+} be as in Proposition 5.10. Let $L_+^s(\varepsilon \otimes \rho)$ be the oriented Lagrangian reduction for the flat $\varepsilon \otimes \rho$ -bundle over $T^1\Sigma$. Suppose that g_l and g_r are trivializations of the restrictions of $L_+^s(\varepsilon \otimes \rho)$ to $T^1\Sigma|_{\Sigma_l}$ and $T^1\Sigma|_{\Sigma_r}$.*

Then $g_l \circ g_r^{-1} : T^1\Sigma|_\gamma \times \mathbf{R}^2 \rightarrow T^1\Sigma|_\gamma \times \mathbf{R}^2$ is a gauge transformation of the trivial oriented \mathbf{R}^2 -bundle over $T^1\Sigma|_\gamma$ and defines a map $h : T^1\Sigma|_\gamma \rightarrow \text{GL}^+(2, \mathbf{R})$. Let $h_ \in \text{Hom}(\pi_1(T^1\Sigma|_\gamma), \pi_1(\text{GL}^+(2, \mathbf{R})))$ denote the map induced by h at the level of fundamental groups.*

- (i) *The image of $h_* \in \text{Hom}(\pi_1(T^1\Sigma|_\gamma), \pi_1(\text{GL}^+(2, \mathbf{R}))) \cong H^1(T^1\Sigma|_\gamma; \mathbf{Z})$ under the connection homomorphism $H^1(T^1\Sigma|_\gamma; \mathbf{Z}) \rightarrow H^2(T^1\Sigma; \mathbf{Z})$ is the Euler class: $\delta(h_*) = e_{\gamma, L_{0+}}(\varepsilon \otimes \rho)$.*
- (ii) *Under the identification $H^1(T^1\Sigma; \mathbf{Z}) \cong \mathbf{Z} \times \mathbf{Z}$ in (5.20), h_* is equal to $(1, 0)$, up to the image of the map $H^1(T^1\Sigma|_{\Sigma_l}; \mathbf{Z}) \oplus H^1(T^1\Sigma|_{\Sigma_r}; \mathbf{Z}) \rightarrow H^1(T^1\Sigma|_\gamma; \mathbf{Z})$.*

Remark 5.22. The ambiguity given by the image of the map $H^1(T^1\Sigma|_{\Sigma_l}; \mathbf{Z}) \oplus H^1(T^1\Sigma|_{\Sigma_r}; \mathbf{Z}) \rightarrow H^1(T^1\Sigma|_\gamma; \mathbf{Z})$ accounts for changing the trivializations g_l and g_r of the restrictions of $L_+^s(\varepsilon \otimes \rho)$ to $T^1\Sigma|_{\Sigma_l}$ and $T^1\Sigma|_{\Sigma_r}$.

Proof of Proposition 5.13. With the notation of Proposition 5.21 $\delta(h_*) = e_{\gamma, L_{0+}}(\varepsilon \otimes \rho)$. Since h_* is equal to $(1, 0)$ in the identification (5.20), Proposition B.3 precisely says that $\delta(h_*) = (2g(\Sigma_l) - 1)[\Sigma] = -\chi(\Sigma_l)[\Sigma]$. \square

Proof of Proposition 5.21. By Proposition 5.14 the restrictions of $L_+^s(\varepsilon \otimes \rho)$ to $T^1\Sigma|_{\Sigma_l}$ and $T^1\Sigma|_{\Sigma_r}$ are indeed trivial.

The first statement $\delta(h_*) = e_{\gamma, L_{0+}}(\varepsilon \otimes \rho)$ follows from the fact that the Euler class is the obstruction to trivialize the oriented Lagrangian bundle $L_+^s(\varepsilon \otimes \rho)$ over $T^1\Sigma$ (see Proposition B.7).

For the second statement, let $T_x^1\Sigma$ and γ be the two generators of the fundamental group of the 2-torus $T^1\Sigma|_\gamma$. We have to show, identifying $\pi_1(\text{GL}^+(2, \mathbf{R}))$ with \mathbf{Z} , that

$$h_*(T_x^1\Sigma) = 1 \text{ and } h_*(\gamma) = 0.$$

In both cases our strategy will be the same: we write the homotopy class we want to calculate as a product of two homotopy classes, the first depending only on the restriction to Σ_l and the second depending on the restriction to Σ_r . With this we can deform the representations and the fiber bundles independently on Σ_l and on Σ_r , without changing the homotopy classes we are considering. By construction of the hybrid representations this means that we can reduce the calculation of the homotopy class to the case when the representations are the restrictions of the irreducible Fuchsian representation for Σ_l and the diagonal Fuchsian representation for Σ_r .

We start by proving the second equality: $h_*(\gamma) = 0$.

We identify γ with $\mathbf{Z} \backslash \mathbf{R}$ so that the restriction of the symplectic bundle to γ is identified with $\mathbf{Z} \backslash (\mathbf{R} \times \mathbf{R}^4)$ where \mathbf{Z} acts diagonally on $\mathbf{R} \times \mathbf{R}^4$, $n \cdot (t, v) = (n+t, (A_\gamma)^n v)$, with $A_\gamma = \rho(\gamma)$ being the diagonal element $\text{diag}(e^{l_\gamma}, e^{k_\gamma}, e^{-l_\gamma}, e^{-k_\gamma})$. Furthermore, the restriction of the oriented Lagrangian reduction $L_+^s(\varepsilon \otimes \rho)$ to the geodesic γ is flat and hence of the form $\mathbf{Z} \backslash (\mathbf{R} \times L_{0+})$.

A trivialization of the symplectic bundle over γ is given by an equivariant map $H : \mathbf{R} \rightarrow \mathrm{Sp}(4, \mathbf{R})$, i.e. $H(t+1) = H(t)A_\gamma^{-1}$ (see Remark 5.16). Such a trivialization induces a trivialization of the Lagrangian bundle $L_+^s(\varepsilon \otimes \rho)|_\gamma$ if furthermore for all t the element $H(t)$ stabilizes L_{0+} . We now provide a “canonical” trivialization in our situation. For this let $A = \mathrm{diag}(e^l, e^k, e^{-l}, e^{-k})$ be a diagonal element and let us denote by L_A the Lagrangian bundle $\mathbf{Z} \backslash (\mathbf{R} \times L_{0+})$ where $n \cdot (t, v) = (n+t, A^n v)$. Then the continuous map

$$H(A, \cdot) : \mathbf{R} \longrightarrow \mathrm{Sp}(4, \mathbf{R}), t \longmapsto H(A, t) = \mathrm{diag}(e^{-tl}, e^{-tk}, e^{tl}, e^{tk})$$

provides a trivialization of L_A .

The homotopy class $h_*(\gamma)$ we are calculating comes from two trivializations $g_l|_\gamma, g_r|_\gamma : L_+^s(\varepsilon \otimes \rho)|_\gamma \rightarrow \gamma \times L_{0+}$, and

$$g_l|_\gamma \circ (g_r|_\gamma)^{-1}(t, v) = (t, h(\gamma(t))v),$$

for $(t, v) \in \gamma \times L_{0+}$.

With the trivialization $H(A_\gamma, \cdot)$ of $L_+^s(\varepsilon \otimes \rho)|_\gamma$ given above, we have

$$g_\star|_\gamma \circ H(A_\gamma, \cdot)^{-1}(t, v) = (t, M_\star(t)v), \text{ for } \star = l, r.$$

This means that in $\pi_1(\mathrm{GL}^+(2, \mathbf{R})) = \mathbf{Z}$ we have the equality

$$h_*(\gamma) = [M_l] - [M_r].$$

We now prove that both $[M_l]$ and $[M_r]$ are trivial.

By construction of the hybrid representation $\rho = \rho_l * \rho_r$ in Section 3.3.1 we know that $A_\gamma = \rho_l(\gamma) = \rho_{irr}(\gamma) = \phi_{irr} \circ \iota(\gamma)$, where $\iota : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ is a discrete embedding and $\phi_{irr} : \mathrm{SL}(2, \mathbf{R}) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ is the irreducible representation. The trivialization given in the proof of Lemma 5.15 and the formula for $H(\cdot, \cdot)$ imply that M_l is (homotopic to) the constant map, thus $[M_l] = 0$.

To compute $[M_r] = 0$ we have to consider $A_\gamma = \rho_r(\gamma) = \rho_{r,1}(\gamma)$, where $(\rho_{r,s})_{s \in [0,1]}$ is a continuous path of maximal representations with $\rho_{r,0} = \rho_\Delta = \phi_\Delta \circ \iota$ and, for all s , $\rho_{r,s}(\gamma)$ is diagonal and where ι is as above and $\phi_\Delta : \mathrm{SL}(2, \mathbf{R}) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ is the diagonal embedding. Thus the family of changes of trivializations $g_{r,s} \circ H(\rho_{r,s}(\gamma), \cdot)^{-1}$, $s \in [0, 1]$, provide a homotopy from the loop $M_r = M_{r,1}$ to the loop $M_{r,0}$, which is the constant map. Therefore $[M_r] = 0$.

We now turn to the proof of the first equality: $h_*(T_x^1 \Sigma) = 1$.

In contrast to the previous calculation, here no equivariance properties are to be satisfied, but the trivializations we choose will not be that natural.

We identify $T_x^1 \Sigma$ with the group $\mathrm{PSO}(2)$, via $T^1 \tilde{\Sigma} \cong \mathrm{PSL}(2, \mathbf{R})$, and identify it also with the boundary $\partial \pi_1(\Sigma)$, via the projection $T^1 \tilde{\Sigma} \cong \partial \pi_1(\Sigma)^{(3+)} \rightarrow \partial \pi_1(\Sigma)$, $(t^s, t, t^u) \mapsto t^s$. We can suppose that under these identifications the attractive fixed point t_γ^s of γ is sent to $[\mathrm{Id}_2]$ in $\mathrm{PSO}(2)$ whereas the repulsive fixed point t_γ^u is sent to $[J] = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right]$.

Since we are working with the representation $\varepsilon \otimes \rho$, the flat $\mathrm{Sp}(4, \mathbf{R})$ -bundle over $T_x^1 \Sigma \cong \mathrm{PSO}(2)$ is the quotient

$$\mathrm{SO}(2) \times_{\{\pm 1\}} \mathrm{Sp}(4, \mathbf{R}) = \{\pm 1\} \backslash (\mathrm{SO}(2) \times \mathrm{Sp}(4, \mathbf{R}))$$

of the trivial bundle over $\mathrm{SO}(2)$ by the group $\{\pm 1\}$, where the action is given by

$$(-1) \cdot (s, g) = (-s, -g).$$

The oriented Lagrangian reduction $L_+^s(\varepsilon \otimes \rho)|_{\text{PSO}(2)}$ is given by the positive continuous curve associated with ρ :

$$\text{SO}(2) \rightarrow \text{PSO}(2) \cong \partial\pi_1(\Sigma) \xrightarrow{\xi_+} \mathcal{L}_+$$

into the space of oriented Lagrangians.

A trivialization of the bundle $\text{SO}(2) \times_{\{\pm 1\}} \text{Sp}(4, \mathbf{R})$ is then a $\{\pm 1\}$ -equivariant map

$$g : \text{SO}(2) \longrightarrow \text{Sp}(4, \mathbf{R}).$$

This trivialization induces furthermore a trivialization of the Lagrangian reduction $L_+^s(\varepsilon \otimes \rho)|_{\text{PSO}(2)}$ if for all α in $\text{SO}(2)$

$$g(\alpha) \cdot \xi_+(\alpha) = L_{0+}^s.$$

We observe that

$$\xi_+(\text{Id}_2) = L_{0+}^s \text{ and } \xi_+ \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) = \xi_+(J) = L_{0+}^u,$$

where $L_{0+}^s = \langle e_1, e_2 \rangle$ and $L_{0+}^u = \langle e_3, e_4 \rangle$ with e_1, \dots, e_4 the standard symplectic basis of \mathbf{R}^4 .

Lemma 5.23. *Let $\eta : \text{SO}(2) \rightarrow \mathcal{L}_+$ be a continuous, $\{\pm 1\}$ -invariant, positive curve. Hence η defines a Lagrangian reduction L_η of the bundle $\text{SO}(2) \times_{\{\pm 1\}} \text{Sp}(4, \mathbf{R})$.*

Suppose that

$$\eta(\text{Id}_2) = L_{0+}^s \text{ and } \eta(J) = L_{0+}^u.$$

Then for all α , $\eta(\alpha)$ and $\phi_\Delta(\alpha) \cdot L_{0+}^u$ are transverse Lagrangians. This means that there exists a unique symmetric 2 by 2 matrix $M(\alpha)$ such that:

$$\begin{pmatrix} \text{Id}_2 & 0 \\ M(\alpha) & \text{Id}_2 \end{pmatrix} \phi_\Delta(\alpha)^{-1} \cdot \eta(\alpha) = L_{0+}^s.$$

The map

$$\beta_\eta : \text{SO}(2) \longrightarrow \text{Sp}(4, \mathbf{R}), \alpha \longmapsto \begin{pmatrix} \text{Id}_2 & 0 \\ M(\alpha) & \text{Id}_2 \end{pmatrix} \phi_\Delta(\alpha)^{-1}$$

is a trivialization of L_η .

Proof. The only point to prove is that $\eta(\alpha)$ and $\phi_\Delta(\alpha) \cdot L_{0+}^u$ are transverse. This is immediate for $\alpha = \pm \text{Id}_2$ and for $\alpha = \pm J$; for other α , for example when (Id_2, α, J) is positively oriented, the positivity of the triples $(L_{0+}^s, \eta(\alpha), L_{0+}^u)$ and $(L_{0+}^u, \phi_\Delta(\alpha) \cdot L_{0+}^s, L_{0+}^s)$ implies that $\eta(\alpha)$ is the graph of $f : L_{0+}^u \rightarrow L_{0+}^s$ with $\omega(\cdot, f \cdot)$ positive definite and that $\phi_\Delta(\alpha) \cdot L_{0+}^u$ is the graph of $g : L_{0+}^u \rightarrow L_{0+}^s$ with $\omega(\cdot, g \cdot)$ negative definite. Since $\omega(v, fv) = \omega(v, gv)$ implies $v = 0$, the transversality of $\eta(\alpha)$ and $\phi_\Delta(\alpha) \cdot L_{0+}^u$ follows. \square

Going back to the proof of Proposition 5.21, the trivializations β_η enable us to write $h_*(T_x^1 \Sigma)$ as the difference:

$$h_*(T_x^1 \Sigma) = [N_l] - [N_r],$$

where, for $\star = l, r$, N_\star is defined as the change of trivializations $g_\star|_{T_x^1 \Sigma} \circ \beta_{\xi_+}^{-1}$.

Again N_l and N_r are in fact homotopic to the corresponding changes of trivializations we obtain from the representations $\phi_{irr} \circ \iota$ and $\phi_\Delta \circ \iota$ respectively. It is then

immediate that N_r is homotopic to the constant map and that N_l is homotopic to the map

$$\begin{aligned} \mathrm{PSO}(2) &\longrightarrow \mathrm{Stab}(L_{0+}^s) \\ [\alpha] &\longmapsto \begin{pmatrix} \mathrm{Id}_2 & 0 \\ -M(\alpha) & \mathrm{Id}_2 \end{pmatrix} \phi_\Delta(\alpha)^{-1} \phi_{irr}(\alpha), \end{aligned}$$

where $M(\alpha)$ is the only 2×2 symmetric matrix such that this product belongs to $\mathrm{Stab}(L_{0+}^s)$. A direct calculation with the formulas given in Facts 3.2.(ii) and 3.3.(ii)

gives for $\alpha = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

$$\begin{pmatrix} \mathrm{Id}_2 & 0 \\ -M(\alpha) & \mathrm{Id}_2 \end{pmatrix} \phi_\Delta(\alpha)^{-1} \phi_{irr}(\alpha) = \begin{pmatrix} A(\theta) & * \\ 0 & {}^tA(\theta)^{-1} \end{pmatrix},$$

where

$$A(\theta) = \frac{4}{3 + \cos^2(2\theta)} \begin{pmatrix} \cos(2\theta) & -\frac{\sqrt{3}}{2} \sin(2\theta) \\ \frac{\sqrt{3}}{2} \sin(2\theta) & \cos(2\theta) \end{pmatrix}.$$

Hence a path in $\mathrm{GL}^+(2, \mathbf{R})$ representing N_l is $[0, \pi] \rightarrow \mathrm{GL}^+(2, \mathbf{R})$, $\theta \mapsto A(\theta)$. It follows that $h_*(T_x^1 \Sigma) = 1$. \square

Remark 5.24. The calculation for negatively adjusted pairs would amount to conjugating the map h by $\mathrm{diag}(1, 1, -1, -1)$ hence to changing h_* in $-h_*$. This leads to the announced value for the Euler class (Remark 5.12).

Remark 5.25. The above computation of the Euler class for hybrid representations hints towards a more general gluing formula for topological invariants of representations for surfaces with boundary.

5.6. Zariski density. Here we prove Theorem 15 of the introduction. First we state a lemma describing the possible Zariski closures of maximal representations.

Lemma 5.26. *Let $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ be a maximal representation. Then the identity component of the Zariski closure of $\rho(\pi_1(\Sigma))$ is (up to conjugation)*

- $\mathrm{Sp}(4, \mathbf{R})$,
- $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$,
- the diagonal $\mathrm{SL}(2, \mathbf{R})$,
- $\mathrm{SL}(2, \mathbf{R}) \times_{\{\pm 1\}} \mathrm{SO}(2)$, the product of the diagonal $\mathrm{SL}(2, \mathbf{R})$ and the identity component of its centralizer,
- or the irreducible $\mathrm{SL}(2, \mathbf{R})$.

Remark 5.27. A similar statement can be found in [9, Prop. 4.8].

Proof. This proof uses basic terminology and concepts in the theory of Lie groups as available in [35].

Let L be the identity component of the Zariski closure of $\rho(\pi_1(\Sigma))$ and suppose that $L \neq G = \mathrm{Sp}(4, \mathbf{R})$. Up to taking a finite cover one can suppose $\rho(\pi_1(\Sigma)) \subset L$. The Toledo number $\tau(\rho)$ is the image under the map $\pi_1(L) \rightarrow \pi_1(G) = \mathbf{Z}$ of the obstruction class $o(\rho)$ of $\rho : \pi_1(\Sigma) \rightarrow L$. The group L is the semidirect product $R \ltimes U$ of its unipotent radical U and a reductive group $R < G$. Since U is contractible, $\pi_1(L) = \pi_1(R)$ and the obstruction class $o(\rho_R)$ of the representation $\rho_R : \pi_1(\Sigma) \rightarrow R = L/U$ equals $o(\rho)$.

The reductive group R is the almost product of its center Z and a semisimple group S , i.e. $R = Z \cdot S$. Let S_c the product of the simple compact factors of S and S_{nc} the product of the noncompact ones so that S is the almost product of S_{nc} and S_c , i.e. $S = S_{nc} \cdot S_c$.

Again, up to taking a finite cover we can suppose that ρ_R lifts to a representation $(\rho_Z, \rho_{nc}, \rho_c) : \pi_1(\Sigma) \rightarrow Z \times S_{nc} \times S_c$ and that Z is connected. Hence the Toledo number $\tau(\rho)$ is the image of the obstruction $(o(\rho_Z), o(\rho_{nc}), o(\rho_c))$ under the map $\pi_1(Z) \times \pi_1(S_{nc}) \times \pi_1(S_c) \rightarrow \pi_1(G)$ induced by the multiplication map $Z \times S_{nc} \times S_c \rightarrow G$. Since Z is connected and abelian, $\text{Hom}(\pi_1(\Sigma), Z) \cong Z^{2g}$ is connected and $o(\rho_Z) = 0$. Since $\pi_1(S_c)$ is finite, its image in $\pi_1(G) \cong \mathbf{Z}$ is $\{0\}$. These last remarks imply that $\tau(\rho)$ is the image of $o(\rho_{nc})$ under $\pi_1(S_{nc}) \rightarrow \pi_1(G)$.

Since $\tau(\rho_{nc}) \neq 0$ it follows that the abelian group $\pi_1(S_{nc})$ has a \mathbf{Z} factor. The real rank of S_{nc} must be 1 or 2 since it is obviously bounded by the rank of G .

Suppose first that the rank of S_{nc} is 1. The restriction on the fundamental group and classification give that S_{nc} is a cover of $\text{PU}(1, n)$. Since the dimension of $\mathfrak{g} = \mathfrak{sp}(4, \mathbf{R})$ is 10 we have $n \leq 2$. If the 8-dimensional Lie algebra $\mathfrak{su}(1, 2)$ were a subalgebra of the 10-dimensional Lie algebra \mathfrak{g} , this would give a $\mathfrak{su}(1, 2)$ -module of dimension $2 = 10 - 8$ hence a morphism $\mathfrak{su}(1, 2) \rightarrow \mathfrak{gl}(2, \mathbf{R})$, this is impossible. Hence $\mathfrak{s}_{nc} \cong \mathfrak{su}(1, 1) \cong \mathfrak{sl}(2, \mathbf{R})$ and S_{nc} is a finite cover of $\text{PSL}(2, \mathbf{R})$.

Embeddings of $\mathfrak{sl}(2, \mathbf{R})$ into \mathfrak{g} are in correspondence with $\mathfrak{sl}(2, \mathbf{R})$ -modules of dimension 4 together with an invariant symplectic form. Their list is the following⁶

- (i) $(V_4, \omega_4), (V_2, \omega_2) \oplus (V_2, \omega_2)$
- (ii) $(V_2, \omega_2) \oplus V_1 \oplus V_1^*$
- (iii) $V_2 \oplus V_2^*, V_1 \oplus V_1^* \oplus V_1 \oplus V_1^*$.

Note that in any case there is a corresponding morphism $\text{SL}(2, \mathbf{R}) \rightarrow G$ so that (up to taking a finite cover of Σ) one can always suppose that $S_{nc} = \text{SL}(2, \mathbf{R})$. The obstruction $o(\rho_{nc})$ is an integer bounded in absolute value by $g - 1$ by Milnor-Wood inequality [40, Th. 1]. However in the above list, the modules in (i) induce multiplication by 2 from $\pi_1(\text{SL}(2, \mathbf{R})) \cong \mathbf{Z}$ to $\pi_1(G) \cong \mathbf{Z}$, the module in (ii) gives the identity $\mathbf{Z} \rightarrow \mathbf{Z}$ and the module in (iii) the zero map. Since $\tau(\rho)$ is $2(g - 1)$ this shows that $o(\rho_{nc}) = g - 1$ and that the embedding $S_{nc} \rightarrow G$ is induced by one of the modules in (i).

It remains to analyze what $\mathfrak{l} = \text{Lie}(L)$ could be. Clearly it is a sub- \mathfrak{s}_{nc} -module of \mathfrak{g} so we first need to describe \mathfrak{g} as a $\mathfrak{sl}(2, \mathbf{R})$ -module. In the first case of (i), one easily find $\mathfrak{g} = V_3 \oplus V_7 = \mathfrak{s}_{nc} \oplus V_7$ and hence $\mathfrak{l} = \mathfrak{r} = \mathfrak{s}_{nc}$ and L is the irreducible $\text{SL}(2, \mathbf{R})$. In the second case of (i), $\mathfrak{g} = V_3 \oplus V_1 \oplus V_3 \oplus V_3 = \mathfrak{s}_{nc} \oplus \mathfrak{z} \oplus V_3 \oplus V_3$ where $\mathfrak{z} \cong \mathfrak{so}(2)$ is the centralizer of \mathfrak{s}_{nc} in \mathfrak{g} . In this case one easily sees that the only nilpotent subalgebra invariant by \mathfrak{s}_{nc} is \mathfrak{z} , hence $\mathfrak{u} = \{0\}$ and $\mathfrak{l} = \mathfrak{r}$. Since (by construction) $\mathfrak{s}_{nc} \subset \mathfrak{r} \subset \mathfrak{s}_{nc} \oplus \mathfrak{z}$ there are only two possibilities for L : either the diagonal $\text{SL}(2, \mathbf{R})$ or the product of this $\text{SL}(2, \mathbf{R})$ and the identity component of its centralizer, i.e. $L = \text{SL}(2, \mathbf{R}) \times_{\{\pm 1\}} \text{SO}(2)$.

We turn now to the case where S_{nc} is of rank 2. If S_{nc} were simple and a proper subgroup of $\text{Sp}(4, \mathbf{R})$ then it had to be isomorphic to $\text{SL}(3, \mathbf{R})$. This would imply that $\tau(\rho) = 0$. Thus S_{nc} is a product $S_1 \times S_2$. Again by a dimension count

⁶Here V_n denote the irreducible $\mathfrak{sl}(2, \mathbf{R})$ -module of dimension n , V_{2k} has an invariant symplectic form ω_{2k} and for any vector space V , $V \oplus V^*$ has a natural symplectic form given by the pairing between V and V^* . Hence the two modules $(V_2, \omega_2) \oplus (V_2, \omega_2)$ and $V_2 \oplus V_2^*$ although isomorphic as $\mathfrak{sl}(2, \mathbf{R})$ -modules are not isomorphic *symplectic* $\mathfrak{sl}(2, \mathbf{R})$ -modules.

$\mathfrak{s}_1 \cong \mathfrak{s}_2 \cong \mathfrak{sl}(2, \mathbf{R})$. Since the centralizer of \mathfrak{s}_1 in \mathfrak{g} is big, inspecting the above list (i)-(iii) implies that the only possible structure of \mathfrak{s}_1 -module on \mathbf{R}^4 is $(V_2, \omega_2) \oplus V_1 \oplus V_1^*$. The similar observation holds also for \mathfrak{s}_2 . Consequently the embedding $\mathfrak{s}_1 \times \mathfrak{s}_2 \subset \mathfrak{g}$ is the embedding associated to a decomposition of $\mathbf{R}^4 = V \oplus W$ into two symplectic planes:

$$\mathfrak{s}_1 \times \mathfrak{s}_2 \cong \mathfrak{sp}(V) \times \mathfrak{sp}(W) \subset \mathfrak{sp}(V \oplus W) \cong \mathfrak{g}.$$

The decomposition of \mathfrak{g} as a \mathfrak{s}_{nc} -module is now $\mathfrak{g} = \mathfrak{s}_{nc} \oplus V \otimes W$ and the equality $\mathfrak{l} = \mathfrak{s}_{nc}$ follows. Hence L is the group $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R}) \subset G$. \square

Remark 5.28. As a corollary we find that L is always a reductive group of Hermitian type, that its centralizer is compact and that the embedding $L \rightarrow G$ is tight. This is a special case of a general result for maximal representations [13, Theorem 4]. However this additional a priori knowledge on L would not have simplified the proof of the lemma much. Certainly classification of Lie groups can be avoided.

Proof of Theorem 15. Let ρ in $\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ be a representation such that $e_\gamma(\rho) \neq g(\Sigma) - 1$. If $\Sigma' \rightarrow \Sigma$ is a k -fold cover and γ' is the simple curve above γ , then $e_{\gamma'}(\rho|_{\pi_1(\Sigma')}) = k e_\gamma(\rho)$ and $g(\Sigma') - 1 = k(g(\Sigma) - 1)$ so that the assumptions of the theorem are still satisfied for Σ' . Therefore, passing to a finite cover, we can suppose that the Zariski closure L of $\rho(\pi_1(\Sigma))$ is connected.

We now inspect the different possibilities for L given by Lemma 5.26. If L is the irreducible or the diagonal $\mathrm{SL}(2, \mathbf{R})$ then, by Proposition 5.10, we would have $e_\gamma(\rho) = g - 1$. If $L = \mathrm{SL}(2, \mathbf{R}) \times_{\{\pm 1\}} \mathrm{SO}(2)$ then ρ could be deformed to a representation ρ' into the diagonal $\mathrm{SL}(2, \mathbf{R})$ and the Euler class would be $e_\gamma(\rho) = e_\gamma(\rho') = g - 1$. If $L = \mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$ then $\rho = (\rho_1, \rho_2)$ is a pair of maximal representations $\pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$. Connectedness of the Teichmüller space implies that ρ can be deformed to a representation $\rho' = (\rho'_1, \varepsilon \cdot \rho'_1)$ with $\rho'_1 : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ maximal and $\varepsilon : \pi_1(\Sigma) \rightarrow \{\pm 1\}$. Up to taking a cover, ρ' is a representation in the diagonal $\mathrm{SL}(2, \mathbf{R})$ and we would have $e_\gamma(\rho) = e_\gamma(\rho') = g - 1$. Hence the only possibility is $L = \mathrm{Sp}(4, \mathbf{R})$ proving the Zariski density of ρ . \square

6. ACTION OF THE MAPPING CLASS GROUP

This section gives the proof of Theorem 10 of the introduction, see Corollaries 6.3 and 6.5 below.

It is known that the action of the mapping class group $\mathrm{Mod}(\Sigma)$ on

$$\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) = \mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))/\mathrm{Sp}(2n, \mathbf{R})$$

is properly discontinuous [37, Theorem 1.0.2] [45, Theorem 1.1]. Furthermore the mapping class group acts naturally on $H^i(T^1\Sigma; \mathbf{F}_2)$ and the maps

$$sw_i : \mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) \longrightarrow H^i(T^1\Sigma; \mathbf{F}_2)$$

are equivariant.

Therefore, understanding the action of $\mathrm{Mod}(\Sigma)$ on the subspaces of $H^i(T^1\Sigma; \mathbf{F}_2)$ where the Stiefel-Whitney classes take their values in, allows us to determine the number of connected components of the quotient of $\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ by $\mathrm{Mod}(\Sigma)$.

Lemma 6.1. *Let $A = \mathbf{Z}$ or \mathbf{F}_2 . The mapping class group of Σ acts trivially on the image of $H^2(\Sigma; A)$ in $H^2(T^1\Sigma; A)$.*

Proof. This is immediate from the fact that $\mathcal{M}od(\Sigma)$ preserves the image of $H^2(\Sigma; A)$ in $H^2(T^1\Sigma; A)$ and acts trivially on $H^2(\Sigma; A)$. \square

For $H^1(T^1\Sigma; \mathbf{F}_2)$ the picture is a bit more complicated.

Proposition 6.2. *The action of the mapping class group on $H^1(T^1\Sigma; \mathbf{F}_2)$ preserves the two cosets $H^1(\Sigma; \mathbf{F}_2)$ and $H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$.*

- (i) *On $H^1(\Sigma; \mathbf{F}_2)$ the action of the mapping class group has two orbits, $\{0\}$ and $H^1(\Sigma; \mathbf{F}_2) \setminus \{0\}$.*
- (ii) *On $H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$ the action of the mapping class group has two orbits.*

Proof. It is obvious that the mapping class group of Σ preserves the two cosets $H^1(\Sigma; \mathbf{F}_2)$ and $H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$.

On $H^1(\Sigma; \mathbf{F}_2) \cong \mathbf{F}_2^{2g}$ the mapping class group acts by symplectomorphisms and it is a classical fact that it generates the whole group $\mathrm{Sp}(2n, \mathbf{F}_2)$ [15, Prop. 7.3]. This action has two orbits, 0 and $\mathbf{F}_2^{2g} \setminus \{0\}$.

The action of the mapping class on the space of spin structures $H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$ has two orbits [34, Corollary 2]. \square

Corollary 6.3. *Let $n \geq 3$. The space $\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))/\mathcal{M}od(\Sigma)$ has 6 connected components.*

Proof. Let us first consider the Hitchin components $\mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$. This subset is invariant by the action of $\mathcal{M}od(\Sigma)$. The 2^{2g} different Hitchin components are indexed by elements $sw_1^A \in H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$ (see Section 4.6.2). The action of the mapping class group has 2 orbits on $H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$. Therefore the quotient $\mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))/\mathcal{M}od(\Sigma)$ has 2 connected components.

The 2×2^{2g} connected components of

$$\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) \setminus \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$$

are indexed by the first and second Stiefel-Whitney classes. The mapping class group acts trivially on the image of $H^2(\Sigma; \mathbf{F}_2)$ in $H^2(T^1\Sigma; \mathbf{F}_2)$ and has two orbits in the coset of $H^1(\Sigma; \mathbf{F}_2)$ in $H^1(T^1\Sigma; \mathbf{F}_2)$. This implies that it has 4 orbits in the set of n -admissible pairs (Definition 4.19), hence the quotient

$$(\mathrm{Rep}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R})) \setminus \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))) / \mathcal{M}od(\Sigma)$$

has 4 connected components. \square

6.1. The case of $\mathrm{Sp}(4, \mathbf{R})$. To define the Euler class of a representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ with $sw_1(\rho) = 0$ we had to make several choices (see Section 4.4). In particular, we fixed a nontrivial element $\gamma \in \pi_1(\Sigma)$ to define the space $\mathrm{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \gamma, L_{0+}}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ and the Euler class $e_{\gamma, L_{0+}}$. Denoting $\mathrm{Rep}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ the quotient of $\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R}))$ by the action of $\mathrm{Sp}(4, \mathbf{R})$, the equivariance of

$$e_{\gamma, L_{0+}} : \mathrm{Rep}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4, \mathbf{R})) \longrightarrow H^2(T^1\Sigma; \mathbf{Z})$$

only holds for the subgroup $\mathrm{Stab}(\gamma) \subset \mathcal{M}od(\Sigma)$ of the mapping class group stabilizing the homotopy class of γ .

This implies

Proposition 6.4. *Let \mathcal{C} be a component of $\text{Rep}_{\max, sw_1=0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \setminus \text{Rep}_{\text{Hitchin}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$. Then \mathcal{C} is sent to itself by the action of the mapping class group of Σ .*

Proof. It is a classical fact that the mapping class group $\text{Mod}(\Sigma)$ is generated by Dehn twists along simple closed curves (see for example [33, Theorem 4.2.D]). As a consequence $\text{Mod}(\Sigma)$ is generated by $\{\text{Stab}(\gamma)\}_{\gamma \in \pi_1(\Sigma) \setminus \{1\}}$. With this said it is enough to show the invariance of \mathcal{C} under $\text{Stab}(\gamma)$. However we know that $\text{Rep}_{\max, sw_1=0} \setminus \text{Rep}_{\text{Hitchin}}$ is invariant under $\text{Mod}(\Sigma)$ and that

$$\begin{aligned} e_{\gamma, L_{0+}} : \pi_0(\text{Rep}_{\max, sw_1=0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \setminus \text{Rep}_{\text{Hitchin}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))) \\ \longrightarrow H^2(T^1\Sigma; \mathbf{Z})^{\text{tor}} \end{aligned}$$

is a bijection. The equivariance property of $e_{\gamma, L_{0+}}$ with respect to $\text{Stab}(\gamma)$ implies the claim. \square

Corollary 6.5. *The space $\text{Rep}_{\max}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))/\text{Mod}(\Sigma)$ has $2g+2$ connected components.*

Proof. Let us write $\text{Rep}_{\max}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$ as a union of subspaces

$$\begin{aligned} \text{Rep}_{\max}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) = \\ \text{Rep}_{\max, sw_1 \neq 0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \bigcup \text{Rep}_{\text{Hitchin}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \bigcup \\ ((\text{Rep}_{\max, sw_1=0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \setminus \text{Rep}_{\text{Hitchin}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))). \end{aligned}$$

The mapping class group preserves this decomposition. Since $\text{Mod}(\Sigma)$ has two orbits on $H^1(T^1\Sigma; \mathbf{F}_2) \setminus H^1(\Sigma; \mathbf{F}_2)$, the space $\text{Rep}_{\text{Hitchin}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))/\text{Mod}(\Sigma)$ has 2 connected components. The mapping class group acts transitively on $H^1(\Sigma; \mathbf{F}_2) \setminus \{0\}$ and trivially on the image of $H^2(\Sigma; \mathbf{F}_2) \subset H^2(T^1\Sigma; \mathbf{F}_2)$, thus the space

$$\text{Rep}_{\max, sw_1 \neq 0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$$

has 2 connected components. By Proposition 6.4 the mapping class group stabilizes the others components. This gives a total of $2g+2$ connected components. \square

7. HOLONOMY OF MAXIMAL REPRESENTATIONS

In this section we prove Theorem 16 of the introduction.

Let $\rho : \pi_1(\Sigma) \rightarrow \text{Sp}(2n, \mathbf{R})$ be a maximal representation. As already noted in Corollary 2.15 for any nontrivial γ the element $\rho(\gamma)$ is (conjugate to) an element of $\text{GL}(n, \mathbf{R})$ whose eigenvalues are in absolute value bigger than 1. For representations in the Hitchin components, Corollary 2.15 implies moreover that $\rho(\gamma)$ is semi-simple. The following statement shows that we cannot expect anything similar for maximal representations in general.

Theorem 7.1. *Let \mathcal{H} be a connected component of*

$$\text{Rep}_{\max}(\pi_1(\Sigma), \text{Sp}(2n, \mathbf{R})) \setminus \text{Rep}_{\text{Hitchin}}(\pi_1(\Sigma), \text{Sp}(2n, \mathbf{R})),$$

and let γ be an element in $\pi_1(\Sigma) \setminus \{1\}$ corresponding to a simple curve.

If $n=2$, the genus of Σ is 2 and γ is separating, we require that \mathcal{H} is not the connected component determined by $sw_1=0$ and $e_\gamma=0$.

Then there exist

- (i) a representation $\rho \in \mathcal{H}$ such that the Jordan decomposition of $\rho(\gamma)$ in $\mathrm{GL}(n, \mathbf{R})$ has a nontrivial parabolic component.
- (ii) a representation $\rho' \in \mathcal{H}$ such that the Jordan decomposition of $\rho'(\gamma)$ in $\mathrm{GL}(n, \mathbf{R})$ has a nontrivial elliptic component.

We first establish some preliminary results towards the proof of the theorem.

Lemma 7.2. *Let Σ be a surface with one boundary component $\gamma = \partial\Sigma$. Let G be a semisimple Lie group and $\rho_0 : \pi_1(\Sigma) \rightarrow G$ a representation whose centralizer in G is finite.*

Then the differential of the map

$$\begin{array}{ccc} \mathrm{Hom}(\pi_1(\Sigma), G) & \xrightarrow{\tau_\gamma} & G \\ \rho & \longmapsto & \rho(\gamma) \end{array}$$

at the point ρ_0 is surjective.

Proof. One can always find a set $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ freely generating $\pi_1(\Sigma)$ and such that $\gamma = [a_1, b_1] \cdots [a_k, b_k]$. The result is then simply a reformulation of [20, Proposition 3.7]. \square

Lemma 7.3. *Let γ be a simple closed separating curve on a closed surface Σ ; denote by Σ_1 and Σ_2 the components of $\Sigma \setminus \gamma$. Let \mathcal{H} be a component of $\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))$ such that the conditions of Theorem 7.1 are satisfied.*

Then there exists ρ in \mathcal{H} such that

- $\rho(\gamma)$ (considered as an element of $\mathrm{GL}(n, \mathbf{R}) < \mathrm{Sp}(2n, \mathbf{R})$) is a multiple of the identity,
- the restriction of ρ to $\pi_1(\Sigma_1)$ (resp. $\pi_1(\Sigma_2)$) has finite centralizer.

Proof. By Theorem 11 and Theorem 14 we only need to prove that there are representations satisfying the conclusions of the lemma in a neighborhood of a model representation (i.e. a standard maximal representation or a hybrid representation).

First consider a diagonal Fuchsian representation $\rho_0 = \phi_\Delta \circ \iota$. There exist deformations $\iota_{1,t}, \dots, \iota_{n,t}$ ($t \in [0, 1]$) such that $\iota_{i,t}(\gamma) = \iota(\gamma)$ and $\iota_{i,0} = \iota$ and, for all $t > 0$, the representation $\rho_t = (\iota_{1,t}, \dots, \iota_{n,t})$ sends $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$ into a Zariski dense subgroup of $\mathrm{SL}(2, \mathbf{R})^n < \mathrm{Sp}(2n, \mathbf{R})$ (this construction was already used in [12, Section 9]). As the centralizer of $\mathrm{SL}(2, \mathbf{R})^n$ inside $\mathrm{Sp}(2n, \mathbf{R})$ is finite, the statement of the lemma follows.

Now consider a twisted diagonal representation $\rho_0 = \iota \otimes \Theta$ which cannot be deformed to a diagonal Fuchsian representation. By Lemma 5.5 it is sufficient to consider the case when Θ has finite image in $\mathrm{O}(2) < \mathrm{O}(n)$ or in $\mathrm{SO}(3) < \mathrm{SO}(n)$. In the first case, the representation ρ_0 takes values in $\mathrm{Sp}(4, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})^{n-2} < \mathrm{Sp}(2n, \mathbf{R})$; we write $\rho_0 = (\iota \otimes \Theta, \iota, \dots, \iota)$. As above, we can find a deformation $\rho_t = (\iota_{1,t} \otimes \Theta, \iota_{2,t}, \dots, \iota_{n-1,t})$ of ρ_0 such that, for all t , $\rho_t(\gamma) = \rho(\gamma)$ and, for all $t > 0$, the Zariski closure of $\rho_t(\pi_1(\Sigma_1))$ contains $\phi_\Delta(\mathrm{SL}(2, \mathbf{R})) \times \mathrm{SL}(2, \mathbf{R})^{n-2} < \mathrm{Sp}(4, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})^{n-2} < \mathrm{Sp}(2n, \mathbf{R})$. The same is true for $\rho_t(\pi_1(\Sigma_2))$. This already means that the centralizer of $\rho_t(\pi_1(\Sigma_1))$ is contained in $\mathrm{O}(2) < \mathrm{O}(n) < \mathrm{Sp}(2n, \mathbf{R})$; it also implies that the image $\Theta(\pi_1(\Sigma_1))$ is contained in the Zariski closure of $\rho_t(\pi_1(\Sigma_1))$.

Suppose now that the image of $\pi_1(\Sigma_1)$ by Θ is not contained in $\mathrm{SO}(2)$. Then there exists a reflection $R \in \mathrm{O}(2) \setminus \mathrm{SO}(2)$ that belongs to the Zariski closure

$\overline{\rho_t(\pi_1(\Sigma_1))}^{\mathbb{Z}}$. Therefore the centralizer of $\overline{\rho_t(\pi_1(\Sigma_1))}^{\mathbb{Z}}$ (which equals the centralizer of $\rho_t(\pi_1(\Sigma_1))$) will be contained in the centralizer of R in $O(2)$ which is finite.

If the restriction of Θ to $\pi_1(\Sigma_1)$ is contained in $SO(2)$, we can suppose that this restriction is the trivial representation. In this situation we write ρ_0 as amalgamated representation $\rho_0 = \rho_1 * \rho_2$, where $\rho_i = \rho_0|_{\pi_1(\Sigma_i)}$. Then the restriction of Θ to $\pi_1(\Sigma_2)$ is not contained in $SO(2)$ (otherwise ρ_0 would be in the same connected component as a diagonal Fuchsian representation). We then deform ρ_0 as an amalgamated representation: $\rho_t^{(1)} * \rho_t^{(2)}$, where $\rho_t^{(1)}$ is a deformation of ρ_1 considered as a diagonal Fuchsian representation (the first case we investigated) and $\rho_t^{(2)}$ is a deformation of the twisted diagonal representation ρ_2 . The centralizer of $\pi_1(\Sigma_2)$ is finite by the same argument as above.

The case when Θ is in $SO(3)$ with finite image can be treated in a similar way and is left to the reader (observe that, when Θ does not lift to $Spin(3)$, the centralizer of Θ is $\{\pm \text{Id}_3\}$).

Let us now assume that $n = 2$ and ρ_0 is a hybrid representation in $\text{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \setminus \text{Hom}_{\text{Hitchin}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R}))$. The definition of a hybrid representation ρ involves the choice of a subsurface Σ' in Σ , for whose fundamental group we choose an irreducible Fuchsian representation. Since we want the holonomy around γ to be a multiple of the identity in $\text{GL}(2, \mathbf{R})$ the curve γ has to be contained in $\Sigma \setminus \Sigma'$ and not homotopic to a boundary component of Σ' . This requires the Euler characteristic $\chi(\Sigma')$ to be different from $3 - 2g$.

The way hybrid representations are constructed in Section 3.3, and with the hypothesis on γ and Σ' , one can ensure that $\rho(\gamma)$ is a multiple of the identity in $\text{GL}(2, \mathbf{R}) < \text{Sp}(4, \mathbf{R})$ and that the Zariski closure $\overline{\rho(\pi_1(\Sigma_1))}^{\mathbb{Z}}$ (resp. $\overline{\rho(\pi_1(\Sigma_2))}^{\mathbb{Z}}$) contains $\text{SL}(2, \mathbf{R}) \times \text{SL}(2, \mathbf{R})$ or an irreducible $\text{SL}(2, \mathbf{R})$ in $\text{Sp}(4, \mathbf{R})$. Therefore, the hybrid representation ρ satisfies all the desired conclusions. Considering not only hybrid representations constructed from positively adjusted pairs as in Section 3.3.1, but also such constructed from negatively adjusted pairs (Definition 3.10) we get representations with Euler class $g - 1 \pm \chi(\Sigma') \in \mathbf{Z}/(2g - 2)\mathbf{Z}$ (see Proposition 5.10 and Remark 5.12). Varying the subsurface Σ' , every Euler characteristic $\chi(\Sigma')$ different from $3 - 2g$ can be attained, hence, when $g > 2$, by Theorem 7 we obtain representations in any connected component of

$$\text{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})) \setminus \text{Hom}_{\text{Hitchin}}(\pi_1(\Sigma), \text{Sp}(4, \mathbf{R})).$$

Only in the case when $g = 2$ the above construction does not give representations with Euler class $e_\gamma = 0$. \square

Proof of Theorem 7.1. Suppose that γ is a separating curve and let ρ_0 in \mathcal{H} be a representation satisfying the conclusions of Lemma 7.3. Let us denote by Σ_1 and Σ_2 the components of $\Sigma \setminus \gamma$. We call σ_1 and σ_2 the two evaluation maps:

$$\begin{aligned} \sigma_i : \text{Hom}(\pi_1(\Sigma_i), \text{Sp}(2n, \mathbf{R})) &\longrightarrow \text{Sp}(2n, \mathbf{R}) \\ \rho &\longmapsto \rho(\gamma). \end{aligned}$$

The representation space for $\pi_1(\Sigma)$ is the fiber product of the representation space for $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$ over $\text{Sp}(2n, \mathbf{R})$:

$$\begin{aligned} \text{Hom}(\pi_1(\Sigma), \text{Sp}(2n, \mathbf{R})) = \\ \{(\rho_1, \rho_2) \in \text{Hom}(\pi_1(\Sigma_2), \text{Sp}(2n, \mathbf{R})) \times \text{Hom}(\pi_1(\Sigma_2), \text{Sp}(2n, \mathbf{R})) \mid \sigma_1(\rho_1) = \sigma_2(\rho_2)\}. \end{aligned}$$

By Lemma 7.2 the map σ_1 (resp. σ_2) is locally surjective in a neighborhood of $\rho_0|_{\pi_1(\Sigma_1)}$ (resp. $\rho_0|_{\pi_1(\Sigma_2)}$). This implies that the map

$$\begin{aligned} \sigma : \text{Hom}(\pi_1(\Sigma), \text{Sp}(2n, \mathbf{R})) &\longrightarrow \text{Sp}(2n, \mathbf{R}) \\ \rho &\longmapsto \rho(\gamma) \end{aligned}$$

is locally surjective in a neighborhood of ρ_0 . As $\sigma(\rho_0) = \rho_0(\gamma)$ is a multiple of the identity in $\text{GL}(n, \mathbf{R})$ we obtain representations ρ and ρ' in a neighborhood of ρ_0 having the desired properties.

When γ is not separating, the proof follows a similar strategy. Let η a simple closed curve separating on Σ such that $\Sigma \setminus \eta = \Sigma_1 \cup \Sigma_2$ with Σ_1 a once punctured torus containing γ . In \mathcal{H} one finds easily a representation ρ_0 such that $\rho_0(\gamma)$ is a multiple of Id_n in $\text{GL}(n, \mathbf{R})$ and such that $\rho_0|_{\pi_1(\Sigma_2)}$ satisfies the hypothesis of Lemma 7.2. Therefore any small enough deformation of $\rho_0|_{\pi_1(\Sigma_1)}$ can be extended to a deformation of ρ_0 . Since γ can be made an element of a subset freely generating the free group $\pi_1(\Sigma_1)$, the result follows. \square

APPENDIX A. MAXIMAL REPRESENTATIONS

A.1. The space of positive curves. In this section we establish certain connectedness properties of the space of positive curves into the Lagrangian Grassmannian.

We will use the notation from Section 2.2: \mathbf{R}^{2n} is a symplectic vector space, with symplectic basis $(e_i)_{i=1, \dots, 2n}$; $\mathcal{X} \subset \mathcal{L} \times \mathcal{L}$ is the space of pairwise transverse Lagrangian subspaces of \mathbf{R}^{2n} , $L_0^s = \text{Span}(e_i)_{1 \leq i \leq n}$, $L_0^u = \text{Span}(e_i)_{n+1 \leq i \leq 2n}$ are two transverse Lagrangian subspaces of \mathbf{R}^{2n} , $P^s, P^u \subset \text{Sp}(2n, \mathbf{R})$ are their stabilizers. The unipotent radical of P^s is

$$U^s = \left\{ u^s(M) = \begin{pmatrix} \text{Id}_n & M \\ 0 & \text{Id}_n \end{pmatrix} \mid M \in \text{M}(n, \mathbf{R}), {}^t M = M \right\}.$$

A Lagrangian L can be written as $u^s(M) \cdot L_0^u$ for some M if and only if L and L_0^s are transverse, in which case M is uniquely determined by L . The triple of Lagrangians (L_0^s, L, L_0^u) is positive (Definition 2.23) if and only if the symmetric matrix M such that $L = u^s(M) \cdot L_0^u$ is positive definite.

Recall that a curve $\xi : S^1 \rightarrow \mathcal{L}$ is said to be *positive* if it sends every positive triple of S^1 to a positive triple of Lagrangians.

Proposition A.1. *The space \mathcal{P} of continuous and positive curves from S^1 to \mathcal{L} is connected. In fact, fixing two points $x^s \neq x^u$ in S^1 , the fibers of the map*

$$\begin{aligned} \mathcal{P} &\longrightarrow \mathcal{X} \subset \mathcal{L} \times \mathcal{L} \\ \xi &\longmapsto (\xi(x^s), \xi(x^u)) \end{aligned}$$

are contractible.

Proof. Consider the set

$$\mathcal{P}_0 = \{ \xi \in \mathcal{P} \mid \xi(x^s) = L_0^s, \xi(x^u) = L_0^u \}.$$

Since $\text{Sp}(2n, \mathbf{R})$ acts transitively on \mathcal{X} , it is enough to show that \mathcal{P}_0 is contractible. For every ξ in \mathcal{P}_0 and every $x \neq x^s$, $\xi(x)$ and $\xi(x^s)$ are transverse, therefore we can regard \mathcal{P}_0 as a subset of the space of maps from $S^1 \setminus \{x^s\}$ to $\text{Sym}(n, \mathbf{R})$. It is precisely the set of maps $\tilde{\xi} : S^1 \setminus \{x^s\} \rightarrow \text{Sym}(n, \mathbf{R})$ such that

- $\lim_{x \rightarrow x^s} u^s(\tilde{\xi}(x)) \cdot L_0^u = L_0^s$
- $\tilde{\xi}$ is continuous

- for all $x > x'$ the symmetric matrix $\tilde{\xi}(x) - \tilde{\xi}(x')$ is positive definite.
- $\tilde{\xi}(x^u) = 0$.

This set of maps is a convex subset of the space of all maps from $S^1 \setminus \{x^s\}$ into $\text{Sym}(n, \mathbf{R})$ (this follows from the fact that the set of positive definite matrices is convex). The contractibility of \mathcal{P}_0 follows. \square

Proposition A.2. *Let γ be a nontrivial element of $\pi_1(\Sigma)$. Then the space of pairs*

$$\mathcal{P}^\gamma = \{(\rho, \xi) \in \text{Hom}(\langle \gamma \rangle, \text{Sp}(2n, \mathbf{R})) \times \mathcal{P} \mid \xi \text{ is } \rho\text{-equivariant}\}$$

has two connected components. If t_γ^s and t_γ^u are the fixed points of γ in $\partial\pi_1(\Sigma)$, then the connected components are detected by which component of $\text{Stab}(\xi(t_\gamma^s)) \cap \text{Stab}(\xi(t_\gamma^u)) \cong \text{GL}(n, \mathbf{R})$ contains $\rho(\gamma)$.

Proof. There are at least two connected components since the sign of $\det(\rho(\gamma)|_{\xi(t_\gamma^s)})$ varies continuously.

It is sufficient to understand the connected components of the fibers of the map

$$\begin{aligned} \phi : \mathcal{P}^\gamma &\longrightarrow \mathcal{X} \\ (\rho, \xi) &\longmapsto (\xi(t_\gamma^s), \xi(t_\gamma^u)). \end{aligned}$$

Again it is enough to calculate the components of $\mathcal{P}_0^\gamma := \phi^{-1}(L_0^s, L_0^u)$. The points t_γ^s and t_γ^u divide the circle $\partial\pi_1(\Sigma)$ in two intervals I_{su} and I_{us} : they are chosen so that x belongs to I_{su} (respectively I_{us}) if and only if the triple $(t_\gamma^s, x, t_\gamma^u)$ (respectively $(t_\gamma^u, x, t_\gamma^s)$) is positively oriented. These two intervals are homeomorphic to \mathbf{R} and isomorphisms are chosen so that the action of γ is conjugate to $t \mapsto t + 1$ on \mathbf{R} .

It is not difficult to show that a curve $\xi : \partial\pi_1(\Sigma) \rightarrow \mathcal{L}$, such that $\xi(t_\gamma^s) = L_0^s$ and $\xi(t_\gamma^u) = L_0^u$, is positive if and only if the following conditions are satisfied:

- for all x in I_{su} the triple $(L_0^s, \xi(x), L_0^u)$ is positive.
- for all x in I_{us} the triple $(L_0^u, \xi(x), L_0^s)$ is positive.
- the restriction of ξ to I_{su} is positive (i.e. it sends positive triples to positive triples).
- the restriction of ξ to I_{us} is positive.

Therefore we can consider the two intervals I_{su} and I_{us} separately. Using the parametrization by symmetric matrices, it is sufficient to show that the set

$$\begin{aligned} \mathcal{S} = \{ (A, \tilde{\xi}) \in \text{GL}(n, \mathbf{R}) \times C^0(\mathbf{R}, \text{Sym}_{>0}(n, \mathbf{R})) \mid \\ \tilde{\xi}(t+1) = A\tilde{\xi}(t)^t A, \text{ and } \forall s < t, \tilde{\xi}(t) - \tilde{\xi}(s) > 0 \} \end{aligned}$$

has two connected components that are distinguished by the sign of $\det A$. (Note that the ρ -equivariance of ξ guarantees that $\lim_{t \rightarrow \infty} u^s(\tilde{\xi}(t)) \cdot L_0^u = L_0^s$ and $\lim_{t \rightarrow -\infty} u^s(\tilde{\xi}(t)) \cdot L_0^u = L_0^s$). Taking into account the natural action of $\text{GL}(n, \mathbf{R})$ on \mathcal{S} reduces the question to determining the connected components of the subset $\mathcal{S}_0 := \{(A, \tilde{\xi}) \in \mathcal{S} \mid \tilde{\xi}(0) = \text{Id}_n\}$. The map

$$\begin{aligned} \mathcal{S}_0 &\longrightarrow \text{GL}(n, \mathbf{R}) \\ (A, \tilde{\xi}) &\longmapsto A. \end{aligned}$$

has convex, hence contractible fibers. Its image is

$$\{A \in \text{GL}(n, \mathbf{R}) \mid A^t A - \text{Id}_n \in \text{Sym}_{>0}(n, \mathbf{R})\}.$$

Using the Cartan decomposition of $\text{GL}(n, \mathbf{R})$, it is easy to show that this set has precisely two connected components given by the sign of $\det A$. \square

Note that the proof gives the following

Proposition A.3. *Let γ be a nontrivial element of $\pi_1(\Sigma)$ and ρ a representation $\langle \gamma \rangle \rightarrow \mathrm{Sp}(2n, \mathbf{R})$. Then the space*

$$\mathcal{P}^\rho = \{\xi \in \mathcal{P} \mid \xi \text{ is } \rho\text{-equivariant}\}$$

is connected.

A.2. Deforming maximal representations in $\mathrm{SL}(2, \mathbf{R})$. The following fact follows from classical Fricke-Klein theory using Fenchel-Nielsen coordinates.

Lemma A.4. *Let $(\gamma_i)_{i=1, \dots, k}$ be a family of pairwise nonhomotopic simple closed curves on Σ and denote by $\gamma_i \in \pi_1(\Sigma)$ the corresponding elements of the fundamental group. Let $\iota_0 : \pi_1(\Sigma) \rightarrow \mathrm{SL}(2, \mathbf{R})$ a discrete embedding with $\iota_0(\gamma_i) = \varepsilon_i g_{i,0} \begin{pmatrix} e^{\lambda_{i,0}} & 0 \\ 0 & e^{-\lambda_{i,0}} \end{pmatrix} g_{i,0}^{-1}$, $\varepsilon_i \in \{\pm 1\}$, $\lambda_{i,0} \in \mathbf{R} \setminus \{0\}$, $g_{i,0} \in \mathrm{SL}(2, \mathbf{R})$. Let $(\lambda_{i,t})_{t \in [0,1]}$ be continuous paths in $\mathbf{R} \setminus \{0\}$.*

Then there exists a continuous path of discrete embeddings $(\iota_t)_{t \in [0,1]}$, and continuous paths $g_{i,t}$ such that for any $t \in [0,1]$, $\iota_t(\gamma_i) = \varepsilon_i g_{i,t} \begin{pmatrix} e^{\lambda_{i,t}} & 0 \\ 0 & e^{-\lambda_{i,t}} \end{pmatrix} g_{i,t}^{-1}$.

A.3. Twisting representations. In this section we explain the strategy which we used to calculate the topological invariants for maximal representations.

A.3.1. The group $\widehat{\pi_1(\Sigma)}$. We fix a discrete embedding of $\pi_1(\Sigma)$ into $\mathrm{PSL}(2, \mathbf{R})$: $\pi_1(\Sigma) < \mathrm{PSL}(2, \mathbf{R})$. Let $\pi : \mathrm{SL}(2, \mathbf{R}) \rightarrow \mathrm{PSL}(2, \mathbf{R})$ be the projection.

We set $\widehat{\pi_1(\Sigma)} = \pi^{-1}(\pi_1(\Sigma)) \subset \mathrm{SL}(2, \mathbf{R})$. The group $\widehat{\pi_1(\Sigma)}$ is a two-to-one cover of $\pi_1(\Sigma)$, which is isomorphic to $\{\pm 1\} \times \pi_1(\Sigma)$. The isomorphism can be chosen so that it intertwines π with the second projection $\{\pm 1\} \times \pi_1(\Sigma) \rightarrow \pi_1(\Sigma)$. Any choice of such an isomorphism amounts to choosing a lift of $\pi_1(\Sigma) < \mathrm{PSL}(2, \mathbf{R})$ to $\mathrm{SL}(2, \mathbf{R})$; such lifts are in one-to-one correspondence with spin-structures on Σ .

For the rest of this section we fix such an isomorphism $\widehat{\pi_1(\Sigma)} = \{\pm 1\} \times \pi_1(\Sigma)$.

A.3.2. Maximal representation of $\widehat{\pi_1(\Sigma)}$.

Definition A.5. A representation $\widehat{\rho} : \widehat{\pi_1(\Sigma)} = \{\pm 1\} \times \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ is said to be *maximal* if the restriction $\widehat{\rho}|_{\pi_1(\Sigma)}$ is maximal (see Definition 2.17).

The set of maximal representation is denoted by $\mathrm{Hom}_{\max}(\widehat{\pi_1(\Sigma)}, \mathrm{Sp}(4, \mathbf{R}))$.

Let $\widehat{\rho} : \widehat{\pi_1(\Sigma)} \rightarrow \mathrm{Sp}(4, \mathbf{R})$ be a maximal representation and $\varepsilon : \widehat{\pi_1(\Sigma)} \rightarrow \{\pm 1\}$ be any representation, then the representation $\varepsilon \cdot \widehat{\rho}$, defined by $\gamma \mapsto \varepsilon(\gamma)\widehat{\rho}(\gamma)$, is also maximal.

If $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ is a maximal representation, then $\widehat{\rho} = \rho \circ pr_2 : \widehat{\pi_1(\Sigma)} \rightarrow \mathrm{Sp}(4, \mathbf{R})$ is a maximal representation, where $pr_2 : \widehat{\pi_1(\Sigma)} = \{\pm 1\} \times \pi_1(\Sigma) \rightarrow \pi_1(\Sigma)$ denotes the projection onto the second factor.

Since $T^1\Sigma \cong \widehat{\pi_1(\Sigma)} \backslash \mathrm{SL}(2, \mathbf{R})$ the notion of *Anosov representations* and *Anosov reductions* (see Section 2.1) can be easily extended to representation of $\widehat{\pi_1(\Sigma)}$. The following lemma is an immediate consequence of Theorem 2.19.

Lemma A.6. *Every maximal representation $\widehat{\rho} : \widehat{\pi_1(\Sigma)} \rightarrow \mathrm{Sp}(4, \mathbf{R})$ is Anosov.*

Similarly to the discussion in Section 4, the Anosov reduction leads to a map

$$\mathrm{Hom}_{\max}(\widehat{\pi_1(\Sigma)}, \mathrm{Sp}(4, \mathbf{R})) \xrightarrow{sw_1} H^1(T^1\Sigma; \mathbf{F}_2)$$

Fixing a nontorsion element $\widehat{\gamma}$ of $\widehat{\pi_1(\Sigma)}$, we introduce the space

$$\mathrm{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \widehat{\gamma}}(\widehat{\pi_1(\Sigma)}, \mathrm{Sp}(4, \mathbf{R}))$$

of pairs (ρ, L_+) consisting of a maximal representation ρ of $\widehat{\pi_1(\Sigma)}$ whose first Stiefel-Whitney class is zero and an attracting oriented Lagrangian L_+ for $\rho(\widehat{\gamma})$. For those pairs, following the discussion in Section 4.4.2, we define an Euler class

$$\mathrm{Hom}_{\max, sw_1=0}^{\mathcal{L}_+, \widehat{\gamma}}(\widehat{\pi_1(\Sigma)}, \mathrm{Sp}(4, \mathbf{R})) \xrightarrow{e_{\widehat{\gamma}}} H^1(T^1\Sigma; \mathbf{Z}).$$

A.3.3. Relations between the invariants. In this section we describe the relations between topological invariants of maximal representations of $\pi_1(\Sigma)$ and $\widehat{\pi_1(\Sigma)}$. More precisely:

- If $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Sp}(4, \mathbf{R})$ is a maximal representation, we compare the invariants of ρ and $\widehat{\rho} = \rho \circ pr_2$.
- If $\widehat{\rho} : \widehat{\pi_1(\Sigma)} \rightarrow \mathrm{Sp}(4, \mathbf{R})$ is a maximal representation and $\varepsilon : \widehat{\pi_1(\Sigma)} \rightarrow \{\pm 1\}$ a homomorphism, we compare the invariants of $\widehat{\rho}$ and $\varepsilon \cdot \widehat{\rho}$.

Lemma A.7. *Let ρ be a maximal representation of $\pi_1(\Sigma)$ and let $\widehat{\rho} = \rho \circ pr_2$. Then*

$$sw_1(\rho) = sw_1(\widehat{\rho}).$$

When $sw_1(\rho) = 0$, let γ be a nontrivial element of $\pi_1(\Sigma)$, $\widehat{\gamma}$ one of the two elements of $\widehat{\pi_1(\Sigma)}$ projecting to γ and let L_+ be an attracting oriented Lagrangian for $\rho(\gamma) = \widehat{\rho}(\widehat{\gamma})$. Then

$$e_{\gamma}(\rho, L_+) = e_{\widehat{\gamma}}(\widehat{\rho}, L_+).$$

Proof. The (oriented) Lagrangian reductions associated with ρ and for $\widehat{\rho}$ are exactly the same, hence their characteristic classes coincide. \square

Lemma A.8. *Let $\widehat{\rho}$ be a maximal representation of $\widehat{\pi_1(\Sigma)}$ and $\varepsilon : \widehat{\pi_1(\Sigma)} \rightarrow \{\pm 1\}$ a representation. Then the first Stiefel-Whitney class of $\widehat{\rho}$ and $\varepsilon \cdot \widehat{\rho}$ coincide:*

$$sw_1(\widehat{\rho}) = sw_1(\varepsilon \cdot \widehat{\rho}).$$

Proof. This lemma follows immediately from Proposition 4.7 since, in this case, the map from $Z = \{\pm \mathrm{Id}\}$ to the group $\pi_0(\mathrm{GL}(2, \mathbf{R}))$ is zero ($-\mathrm{Id}$ is in $\mathrm{GL}^+(2, \mathbf{R})$). \square

Proposition A.9. *Let $\widehat{\rho} : \widehat{\pi_1(\Sigma)} \rightarrow \mathrm{Sp}(4, \mathbf{R})$ be a maximal representation with $sw_1(\widehat{\rho}) = 0$. Let $\widehat{\gamma} \in \widehat{\pi_1(\Sigma)}$ be a nontorsion element and L_+ an attracting oriented Lagrangian for $\widehat{\rho}(\widehat{\gamma})$. Let $\varepsilon : \widehat{\pi_1(\Sigma)} \rightarrow \{\pm 1\}$ be a homomorphism. Then L_+ is an attracting oriented Lagrangian for $(\varepsilon \cdot \widehat{\rho})(\widehat{\gamma})$ and the Euler class (relative to $\widehat{\gamma}$) for the pairs $(\varepsilon \cdot \widehat{\rho}, L_+)$ and $(\widehat{\rho}, L_+)$ are*

$$e_{\widehat{\gamma}}(\varepsilon \cdot \widehat{\rho}, L_+) = e_{\widehat{\gamma}}(\widehat{\rho}, L_+) \in H^2(T^1\Sigma; \mathbf{Z}) \text{ if } \varepsilon(-1) = 1,$$

and

$$e_{\widehat{\gamma}}(\varepsilon \cdot \widehat{\rho}, L_+) = e_{\widehat{\gamma}}(\widehat{\rho}, L_+) + (g-1)[\Sigma] \in H^2(T^1\Sigma; \mathbf{Z}) \text{ if } \varepsilon(-1) = -1.$$

Proof. Let L_1 and L_2 be the Lagrangian reductions associated with $\widehat{\rho}$ and $\varepsilon \cdot \widehat{\rho}$. Denote by L_{1+} and L_{2+} the corresponding oriented Lagrangian bundles determined by the choice of $\widehat{\gamma}$ and L_+ . If D_ε is the flat real line bundle over $T^1\Sigma$ associated with the representation ε , we have

$$L_{2+} = D_\varepsilon \otimes L_{1+}.$$

(Because L_1 has even dimension there is a canonical orientation on $D_\varepsilon \otimes L_{1+}$, even if D_ε is neither oriented nor necessarily orientable)

Let S_1 and S_2 be the associated S^1 -bundles corresponding to L_{1+} and L_{2+} and let S_ε be the flat S^1 -bundle associated with the representation $\varepsilon : \widehat{\pi_1(\Sigma)} \rightarrow \{\pm 1\} \subset S^1$, then the above equality can be restated as

$$S_2 = S_\varepsilon \times_{S^1} S_1.$$

This implies for the Euler classes:

$$e(S_2) = e(S_\varepsilon) + e(S_1).$$

Since $e(S_2) = e_{\widehat{\gamma}}(\varepsilon \cdot \widehat{\rho})$ and $e(S_1) = e_{\widehat{\gamma}}(\widehat{\rho})$, the proposition will follow from the following lemma. \square

Lemma A.10. *Let $\varepsilon : \widehat{\pi_1(\Sigma)} \rightarrow S^1$ be a representation and let S_ε be the associated flat S^1 -bundle. Then*

$$e(S_\varepsilon) = 0 \text{ if } \varepsilon(-1) = 1,$$

and

$$e(S_\varepsilon) = (g-1)[\Sigma] \text{ if } \varepsilon(-1) = -1.$$

Proof. First we note that $e(S_\varepsilon)$ varies continuously with ε . Hence e only depends on the connected component of $\text{Hom}(\widehat{\pi_1(\Sigma)}, S^1)$ containing ε . Since $\text{Hom}(\widehat{\pi_1(\Sigma)}, S^1) = \text{Hom}(\{\pm 1\} \times \pi_1(\Sigma), S^1) = \text{Hom}(\{\pm 1\} \times \mathbf{Z}^{2g}, S^1) \cong \{\pm 1\} \times (S^1)^{2g}$, this space has two connected components distinguished precisely by the value of $\varepsilon(-1)$.

We only need to calculate the Euler class for two specific representations. The first is the trivial representation for which the result is obvious. The second one is the projection $pr_1 : \widehat{\pi_1(\Sigma)} \cong \{\pm 1\} \times \pi_1(\Sigma) \rightarrow \{\pm 1\}$ onto the first factor. Consider the (non-flat) S^1 -bundle over the surface: $\pi_1(\Sigma) \backslash \text{SL}(2, \mathbf{R}) \rightarrow \pi_1(\Sigma) \backslash \mathbb{H}$, its Euler class is given by the Toledo number of the injection $\pi_1(\Sigma) \rightarrow \text{SL}(2, \mathbf{R})$, which is $(g-1)$. One checks that the S^1 -bundle S_ε is the pullback of this S^1 -bundle by the natural projection $\widehat{\pi_1(\Sigma)} \backslash \text{SL}(2, \mathbf{R}) \cong T^1\Sigma \rightarrow \pi_1(\Sigma) \backslash \mathbb{H} \cong \Sigma$. This implies the claim. \square

APPENDIX B. COHOMOLOGY

B.1. The cohomology of $T^1\Sigma$. In this section we compute the cohomology of the unit tangent bundle $T^1\Sigma$ with \mathbf{Z} and \mathbf{F}_2 coefficients and study the connecting homomorphism in the Mayer-Vietoris sequence. The results are used in Section 4.5 and Section 5.5.

Proposition B.1. *Let Σ be a closed, connected, oriented surface of genus $g > 1$. The cohomology groups of $T^1\Sigma$ with \mathbf{Z} coefficients are:*

- $H^0(T^1\Sigma; \mathbf{Z}) = \mathbf{Z}$,
- $H^1(T^1\Sigma; \mathbf{Z}) = \mathbf{Z}^{2g}$,
- $H^2(T^1\Sigma; \mathbf{Z}) = \mathbf{Z}^{2g} \times \mathbf{Z}/(2g-2)\mathbf{Z}$,

$$- H^3(T^1\Sigma; \mathbf{Z}) = \mathbf{Z}.$$

The cohomology groups of $T^1\Sigma$ with \mathbf{F}_2 coefficients are:

$$\begin{aligned} - H^0(T^1\Sigma; \mathbf{F}_2) &= \mathbf{F}_2, \\ - H^1(T^1\Sigma; \mathbf{F}_2) &= \mathbf{F}_2^{2g+1}, \\ - H^2(T^1\Sigma; \mathbf{F}_2) &= \mathbf{F}_2^{2g+1}, \\ - H^3(T^1\Sigma; \mathbf{F}_2) &= \mathbf{F}_2. \end{aligned}$$

Proof. Let A be the ring \mathbf{Z} or \mathbf{F}_2 .

The unit tangent bundle $T^1\Sigma \rightarrow \Sigma$ is a principal S^1 -bundle whose Euler class e is $(2-2g)$ in $\mathbf{Z} \cong H^2(\Sigma; \mathbf{Z})$. The Gysin exact sequence with A -coefficients for this bundle is

$$\begin{aligned} \text{(B.2)} \quad 0 \longrightarrow H^0(\Sigma; A) \longrightarrow H^0(T^1\Sigma; A) \longrightarrow 0 \longrightarrow H^1(\Sigma; A) \\ \longrightarrow H^1(T^1\Sigma; A) \longrightarrow H^0(\Sigma; A) \xrightarrow{\simeq e_A} H^2(\Sigma; A) \longrightarrow H^2(T^1\Sigma; A) \\ \longrightarrow H^1(\Sigma; A) \longrightarrow 0 \longrightarrow H^3(T^1\Sigma; A) \longrightarrow H^2(\Sigma; A) \longrightarrow 0, \end{aligned}$$

where e_A is the image of e under the natural map $H^2(\Sigma; \mathbf{Z}) \rightarrow H^2(\Sigma; A)$. The conclusion for H^0 and H^3 follows immediately from this. When A is \mathbf{Z} , $H^0(\Sigma; \mathbf{Z}) \xrightarrow{\simeq e} H^2(\Sigma; \mathbf{Z})$ is injective and we get the exact sequences:

$$0 \longrightarrow \mathbf{Z}^{2g} \longrightarrow H^1(T^1\Sigma; \mathbf{Z}) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathbf{Z}/(2g-2)\mathbf{Z} \longrightarrow H^2(T^1\Sigma; \mathbf{Z}) \longrightarrow \mathbf{Z}^{2g} \longrightarrow 0.$$

From this the result for H^1 and H^2 follows easily. When A is \mathbf{F}_2 , the connecting map $H^0(\Sigma; A) \xrightarrow{\simeq e_A} H^2(\Sigma; A)$ is zero. \square

The above proof gives a *canonical* isomorphism

$$H^2(T^1\Sigma; \mathbf{Z})^{tor} \cong \mathbf{Z}/(2g-2)\mathbf{Z}$$

between the torsion of $H^2(T^1\Sigma; \mathbf{Z})$ and $\mathbf{Z}/(2g-2)\mathbf{Z}$. In particular, $[\Sigma]$ is the *canonical* generator of $H^2(T^1\Sigma; \mathbf{Z})^{tor}$.

Let γ be a simple closed oriented separating curve on the surface Σ , i.e. $\Sigma \setminus \gamma$ has two connected components, Σ_l denotes the component on the left of γ and Σ_r the component on the right (this uses the orientations of γ and Σ). This induces a decomposition of the unit tangent bundle: $T^1\Sigma$ is the union of $T^1\Sigma|_{\Sigma_l}$ and $T^1\Sigma|_{\Sigma_r}$ identified along $T^1\Sigma|_{\gamma}$. The Mayer-Vietoris sequence for this decomposition reads as

$$\begin{aligned} 0 \longrightarrow H^0(T^1\Sigma; \mathbf{Z}) \longrightarrow H^0(T^1\Sigma|_{\Sigma_l}; \mathbf{Z}) \oplus H^0(T^1\Sigma|_{\Sigma_r}; \mathbf{Z}) \\ \longrightarrow H^0(T^1\Sigma|_{\gamma}; \mathbf{Z}) \longrightarrow H^1(T^1\Sigma; \mathbf{Z}) \longrightarrow H^1(T^1\Sigma|_{\Sigma_l}; \mathbf{Z}) \oplus H^1(T^1\Sigma|_{\Sigma_r}; \mathbf{Z}) \\ \longrightarrow H^1(T^1\Sigma|_{\gamma}; \mathbf{Z}) \longrightarrow H^2(T^1\Sigma; \mathbf{Z}) \longrightarrow H^2(T^1\Sigma|_{\Sigma_l}; \mathbf{Z}) \oplus H^2(T^1\Sigma|_{\Sigma_r}; \mathbf{Z}) \\ \longrightarrow H^2(T^1\Sigma|_{\gamma}; \mathbf{Z}) \longrightarrow H^3(T^1\Sigma; \mathbf{Z}) \longrightarrow 0. \end{aligned}$$

This sequence can also be used to compute the cohomology of the unit tangent bundle. We concentrate on the connecting morphism

$$\delta : H^1(T^1\Sigma|_{\gamma}; \mathbf{Z}) \longrightarrow H^2(T^1\Sigma; \mathbf{Z})$$

and its kernel.

We realize γ as a C^1 loop on Σ , it then has a natural lift to the unit tangent bundle $T^1\Sigma$ which we denote again by γ . This lift induces a trivialization $T^1\Sigma|_\gamma \cong S^1 \times \gamma$ and hence isomorphisms

$$H^1(T^1\Sigma|_\gamma; \mathbf{Z}) \cong H^1(S^1; \mathbf{Z}) \oplus H^1(\gamma; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}.$$

The first identification is the map in cohomology corresponding to the projections $T^1\Sigma|_\gamma \rightarrow S^1$ and $T^1\Sigma|_\gamma \rightarrow \gamma$ whereas the second identification involves the orientations on S^1 and γ (the orientation on $S^1 \cong T_x^1\Sigma$ is induced by the orientation on Σ).

Proposition B.3. *Let γ be an oriented closed simple separating geodesic on the surface Σ .*

Then the orientation class $o_\gamma \in H^1(\gamma; \mathbf{Z}) \cong \mathbf{Z}$ is sent to $[\Sigma]$ by the connecting homomorphism of the Mayer-Vietoris sequence:

$$H^1(\gamma; \mathbf{Z}) \subset H^1(T^1\Sigma|_\gamma; \mathbf{Z}) \xrightarrow{\delta} H^2(T^1\Sigma; \mathbf{Z}).$$

The kernel of δ is generated by the elements:

$$(1, 1 - 2g(\Sigma_l)) \text{ and } (-1, 1 - 2g(\Sigma_r)) \in \mathbf{Z} \times \mathbf{Z} \cong H^1(T^1\Sigma|_\gamma; \mathbf{Z}).$$

Proof. The connecting homomorphisms for the decompositions of the surface and the unit tangent bundle fit in a commutative diagram:

$$\begin{array}{ccc} H^1(\gamma; \mathbf{Z}) & \xrightarrow{\delta} & H^2(\Sigma; \mathbf{Z}) \\ \downarrow & & \downarrow \\ H^1(T^1\Sigma|_\gamma; \mathbf{Z}) & \xrightarrow{\delta} & H^2(T^1\Sigma; \mathbf{Z}). \end{array}$$

So the first result follows from the equality $\delta(o_\gamma) = o_\Sigma$, where o_Σ is the orientation class in $H^2(\Sigma; \mathbf{Z})$. This equality is easy to establish. In fact the Mayer-Vietoris sequence for the surface:

$$H^1(\gamma; \mathbf{Z}) \longrightarrow H^2(\Sigma; \mathbf{Z}) \longrightarrow H^2(\Sigma_l; \mathbf{Z}) \oplus H^2(\Sigma_r; \mathbf{Z})$$

already shows that $\delta : H^1(\gamma; \mathbf{Z}) \rightarrow H^2(\Sigma; \mathbf{Z})$ is surjective so that $\delta(o_\gamma) = \pm o_\Sigma$. The sign conventions are precisely arranged so that $\delta(o_\gamma) = o_\Sigma$.

Due to the exactness of the Mayer-Vietoris sequence the kernel of δ is the image of

$$H^1(T^1\Sigma|_{\Sigma_l}; \mathbf{Z}) \oplus H^1(T^1\Sigma|_{\Sigma_r}; \mathbf{Z}) \longrightarrow H^1(T^1\Sigma|_\gamma; \mathbf{Z}).$$

It is therefore enough to show that the image of

$$(B.4) \quad H^1(T^1\Sigma|_{\Sigma_l}; \mathbf{Z}) \longrightarrow H^1(T^1\Sigma|_\gamma; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$$

is generated by $(1, 1 - 2g(\Sigma_l))$. (The calculation for Σ_r is similar). The commutative square

$$\begin{array}{ccc} H^1(\Sigma_l; \mathbf{Z}) & \longrightarrow & H^1(\gamma; \mathbf{Z}) \\ \downarrow & & \downarrow \\ H^1(T^1\Sigma|_{\Sigma_l}; \mathbf{Z}) & \longrightarrow & H^1(T^1\Sigma|_\gamma; \mathbf{Z}) \end{array}$$

implies that the composition $H^1(\Sigma_l; \mathbf{Z}) \subset H^1(T^1\Sigma|_{\Sigma_l}; \mathbf{Z}) \rightarrow H^1(T^1\Sigma|_\gamma; \mathbf{Z})$ is zero, because the map $H^1(\Sigma_l; \mathbf{Z}) \rightarrow H^1(\gamma; \mathbf{Z})$ is zero as γ is a boundary in Σ_l . The restriction $T^1\Sigma|_{\Sigma_l}$ is the trivial bundle $S^1 \times \Sigma_l$ so that

$$H^1(T^1\Sigma|_{\Sigma_l}; \mathbf{Z}) \cong H^1(S^1; \mathbf{Z}) \times H^1(\Sigma_l; \mathbf{Z}).$$

This means that the image of the above map (B.4) has rank 1 and is the image of

$$\mathbf{Z} \cong H^1(S^1; \mathbf{Z}) \longrightarrow H^1(T^1\Sigma|_\gamma; \mathbf{Z}) \cong \mathbf{Z} \times \mathbf{Z}.$$

The first component of this last map is the identity $\mathbf{Z} \cong H^1(S^1; \mathbf{Z}) \rightarrow H^1(S^1; \mathbf{Z}) \cong \mathbf{Z}$ so that the image of (B.4) is generated by $(1, n)$ for some integer n . To calculate this integer n let us consider the closed surface $\overline{\Sigma}_1 = \Sigma_l \cup_\gamma D^2$ obtained by gluing a disk along γ . The genus of $\overline{\Sigma}_1$ is $g(\overline{\Sigma}_1) = g(\Sigma_l)$ and the two S^1 -bundles $T^1\Sigma|_{\Sigma_l}$ and $T^1\overline{\Sigma}_1|_{\Sigma_l}$ are isomorphic. From the Mayer-Vietoris sequence for the decomposition of $T^1\overline{\Sigma}_1$ we get

$$\begin{aligned} H^1(T^1\Sigma|_{\Sigma_l}; \mathbf{Z}) \oplus H^1(T^1\overline{\Sigma}_1|_{D^2}; \mathbf{Z}) &\longrightarrow H^1(T^1\Sigma|_\gamma; \mathbf{Z}) \longrightarrow H^2(T^1\overline{\Sigma}_1; \mathbf{Z}) \\ &\longrightarrow H^2(T^1\Sigma|_{\Sigma_l}; \mathbf{Z}) \oplus H^2(T^1\overline{\Sigma}_1|_{D^2}; \mathbf{Z}) \longrightarrow H^2(T^1\Sigma|_\gamma; \mathbf{Z}). \end{aligned}$$

A small calculation shows that

$$\mathbf{Z} \cong H^1(T^1\overline{\Sigma}_1|_{D^2}; \mathbf{Z}) \rightarrow H^1(T^1\Sigma|_\gamma; \mathbf{Z}) \cong \mathbf{Z} \times \mathbf{Z}$$

sends 1 to $(-1, 1)$. The exact sequence reduces to

$$0 \longrightarrow \mathbf{Z}^2 / \langle (1, n), (-1, 1) \rangle \longrightarrow H^2(T^1\overline{\Sigma}_1; \mathbf{Z}) \longrightarrow \mathbf{Z}^{2g(\Sigma_l)} \longrightarrow 0.$$

This exact sequence implies that the torsion of $H^2(T^1\overline{\Sigma}_1; \mathbf{Z})$ is isomorphic to $\mathbf{Z}/(n+1)\mathbf{Z}$. This torsion being isomorphic to $\mathbf{Z}/(2g(\Sigma_l)-2)\mathbf{Z}$ by Proposition B.1 the value of n is $1 - 2g(\Sigma_l)$ or $2g(\Sigma_l) - 3$.

Doubling Σ_l along γ to obtain a closed surface, a similar argument shows that the torsion of the double of Σ_l is isomorphic to $\mathbf{Z}/(2n)\mathbf{Z}$. Thus the only possible value for n is $1 - 2g(\Sigma_l)$ because the genus of the double of Σ_l is $2g(\Sigma_l)$. \square

Lemma B.5. *Let $\{\eta_i\}$ be curves in Σ whose images generate the homology group $H_1(\Sigma; \mathbf{Z})$; we denote by $f_i : \eta_i \rightarrow \Sigma$ the inclusion. With a slight abuse of notation we write $T^1\Sigma|_{\eta_i}$ for the pulled back circle bundle $f_i^*T^1\Sigma$ and denote again by $f_i : T^1\Sigma|_{\eta_i} \rightarrow T^1\Sigma$ the corresponding map.*

Let c be a class in $H^2(T^1\Sigma; A)$ such that, for all i ,

$$c|_{T^1\Sigma|_{\eta_i}} := f_i^*(c) = 0.$$

Then c belongs to $\text{Im}(H^2(\Sigma; A) \rightarrow H^2(T^1\Sigma; A))$.

Proof. Using the Gysin exact sequence with A coefficients we only need to show that the image a of c in $H^1(\Sigma; A)$ is zero. As $\{\eta_i\}$ generate the homology it will be the case if $f_i^*(a) = 0$ for all i . Observe that the Gysin sequences for $T^1\Sigma$ and for $T^1\Sigma|_{\eta_i}$ fit in a commutative diagram

$$\begin{array}{ccc} H^2(T^1\Sigma; A) & \longrightarrow & H^1(\Sigma; A) \\ \downarrow & & \downarrow \\ H^2(T^1\Sigma|_{\eta_i}; A) & \longrightarrow & H^1(\eta_i; A). \end{array}$$

So the property follows from the hypothesis $f_i^*(c) = 0$. \square

B.2. Stiefel-Whitney classes for tensor products.

Proposition B.6. *Let L a line bundle and W a n -plane bundle over a (para-compact) base B . Then the first and second Stiefel-Whitney classes of the tensor product $L \otimes W$ are:*

$$sw_1(L \otimes W) = nsw_1(L) + sw_1(W)$$

$$sw_2(L \otimes W) = \frac{n(n-1)}{2}sw_1(L) \smile sw_1(L) + (n-1)sw_1(L) \smile sw_1(W) + sw_2(W).$$

Proof. There exists $f : B_1 \rightarrow B$ such that f^*W splits as the sum of n -line bundles and $f^* : H^*(B) \rightarrow H^*(B_1)$ is injective [32, Ch. 16, Prop. 5.2]. Hence, up to pulling back, one can suppose that W is the sum of n line bundles L_1, \dots, L_n . Let η the first Stiefel-Whitney class of L and η_i the first Stiefel-Whitney class of L_i . Then the first Stiefel-Whitney class of $L \otimes L_i$ is $\eta + \eta_i$ ([32, Ch. 16, Th. 3.4]).

Since the total Stiefel-Whitney class is multiplicative under sums of bundles ([32, Ch. 16, Sec. 3.1]) one finds the following formulas for the first and second Stiefel-Whitney classes of W and $L \otimes W$:

$$\begin{aligned} sw_1(W) &= \sum_{i=1, \dots, n} \eta_i, & sw_1(L \otimes W) &= \sum_{i=1, \dots, n} (\eta + \eta_i) \\ sw_2(W) &= \sum_{1 \leq i < j \leq n} \eta_i \smile \eta_j, & sw_2(L \otimes W) &= \sum_{1 \leq i < j \leq n} (\eta + \eta_i) \smile (\eta + \eta_j). \end{aligned}$$

By expanding the sums, the claim follows. \square

B.3. Euler class in Mayer-Vietoris sequence.

Proposition B.7. *Let F be an oriented n -plane bundle over a base B . Suppose that $B = B_1 \cup_C B_2$ with $F|_{B_1}$ and $F|_{B_2}$ being trivial and that Mayer-Vietoris sequences hold for this decomposition of B . Let $h : C \rightarrow \mathrm{GL}^+(n, \mathbf{R})$ be the change of trivializations, $k : C \rightarrow \mathbf{R}^n \setminus \{0\}$ an orbital map associated with h and t in $H^{n-1}(\mathbf{R}^n \setminus \{0\}; \mathbf{Z})$ the generator.*

Then the Euler class of F is $\delta(k^(t))$ where $\delta : H^{n-1}(C; \mathbf{Z}) \rightarrow H^n(B, \mathbf{Z})$ is the connecting morphism in the Mayer-Vietoris sequence and $k^* : H^{n-1}(\mathbf{R}^n \setminus \{0\}; \mathbf{Z}) \rightarrow H^{n-1}(C; \mathbf{Z})$ the map induced by k .*

In particular the image of $k^(t)$ under the natural map from $H^{n-1}(C; \mathbf{Z})$ to $\mathrm{Hom}(H_{n-1}(C; \mathbf{Z}), \mathbf{Z})$ is $k_* : H_{n-1}(C; \mathbf{Z}) \rightarrow H_{n-1}(\mathbf{R}^n \setminus \{0\}; \mathbf{Z}) \cong \mathbf{Z}$.*

Proof. Below every cohomology group is understood with \mathbf{Z} -coefficients.

Let F^0 the complement of the zero section in F . The Thom class u of F is the unique class of $H^n(F, F^0)$ such that u is sent to the generator in $H^n(F_b, F_b \setminus \{0\})$ ($F_b \cong \mathbf{R}^n$ is one fiber of F) and the Euler class e is the image of u under the natural map $H^n(F, F^0) \rightarrow H^n(F) \cong H^n(B)$ [32, Ch. 16, Sec. 7]. We should first construct a class x in $H^{n-1}(F|_C)$ such that $\delta(x) = e$ in $H^n(E) \cong H^n(B)$ and then identify x

with $k^*(t)$. Consider the commutative diagram:

$$\begin{array}{ccccc}
 & & & & H^{n-1}(F|_C) \\
 & & & & \downarrow \\
 & & H^{n-1}(F^0|_{B_1}) \oplus H^{n-1}(F^0|_{B_2}) & \longrightarrow & H^{n-1}(F^0|_C) \\
 & & \downarrow & & \downarrow \\
 H^n(F, F^0) & \longrightarrow & H^n(F|_{B_1}, F^0|_{B_1}) \oplus H^n(F|_{B_2}, F^0|_{B_2}) & \longrightarrow & H^n(F|_C, F^0|_C) \\
 \downarrow & & \downarrow & & \\
 H^n(F) & \longrightarrow & H^n(F|_{B_1}) \oplus H^n(F|_{B_2}) & &
 \end{array}$$

The horizontal lines are Mayer-Vietoris exact sequences whereas the vertical lines are exact sequences of pairs. Each of those sequences arises from a short exact sequences of differential complexes⁷: $0 \rightarrow (C^*(F), d) \rightarrow (C^*(F|_{B_1}), d) \oplus (C^*(F|_{B_2}), d) \rightarrow (C^*(F|_C), d) \rightarrow 0$, etc. In the above diagram, consider the following classes:

$$\begin{array}{ccccc}
 & & & & x \\
 & & & & \downarrow \\
 & & v_1 \oplus v_2 & \xrightarrow{\quad} & w \\
 & & \downarrow & & \downarrow \\
 u & \xrightarrow{\quad} & u_1 \oplus u_2 & \xrightarrow{\quad} & 0 \\
 \downarrow & & \downarrow & & \\
 e & \xrightarrow{\quad} & 0 \oplus 0 & &
 \end{array}$$

The diagram is self-explanatory: $u_1 \oplus u_2$ is the image of u and u_1 and u_2 are the Thom classes of $F|_{B_1}$ and $F|_{B_2}$. Since these bundles are trivial, the corresponding Euler classes are zero and u_1 and u_2 lift to $v_1 \in H^{n-1}(F^0|_{B_1})$ and $v_2 \in H^{n-1}(F^0|_{B_2})$ respectively. The class w in $H^{n-1}(F^0|_C)$ is the image of $v_1 \oplus v_2$. It projects to 0 in $H^n(F|_C, F^0|_C)$. Hence it comes from a class x in $H^{n-1}(F|_C)$.

We claim that $\delta(x) \in H^n(F)$ is the Euler class of F . Let $V_1 \in C^{n-1}(F^0|_{B_1})$ and $V_2 \in C^{n-1}(F^0|_{B_2})$ represent v_1 and v_2 . Let also $\tilde{V}_1 \in C^{n-1}(F|_{B_1})$ and $\tilde{V}_2 \in C^{n-1}(F|_{B_2})$ be cochains extending V_1 and V_2 , i.e. V_1 is the image of \tilde{V}_1 under the surjective map $C^{n-1}(F|_{B_1}) \rightarrow C^{n-1}(F^0|_{B_1})$. Then $U_1 = d\tilde{V}_1$ and $U_2 = d\tilde{V}_2$ are representatives of u_1 and u_2 and a representative of u is $U = U_1 \oplus U_2$ where we identify $C^n(F, F^0)$ with its image in $C^n(F|_{B_1}, F^0|_{B_1}) \oplus C^n(F|_{B_2}, F^0|_{B_2})$. Under the injection $C^n(F, F^0) \rightarrow C^n(F)$ U represents the Euler class e .

A cochain representing w in $C^{n-1}(F^0|_C)$ is $W = V_2|_C - V_1|_C$ (the map from the complexes associated with B_1 , or B_2 , to the complexes associated with C are simply denoted by $A \mapsto A|_C$). A cochain representing x in $C^{n-1}(F|_C)$ is then $X = \tilde{V}_2|_C - \tilde{V}_1|_C$. Hence X is the image of $\tilde{V}_1 \oplus \tilde{V}_2$ by the map $C^{n-1}(F|_{B_1}) \oplus C^{n-1}(F|_{B_2}) \rightarrow C^{n-1}(F|_C)$. Thus the element $d\tilde{V}_1 \oplus d\tilde{V}_2$ in $C^n(F|_{B_1}) \oplus C^n(F|_{B_2})$ lies in the image of the injective map $C^n(F) \rightarrow C^n(F|_{B_1}) \oplus C^n(F|_{B_2})$. By construction of the connecting morphism, $\delta(x)$ is represented by $d\tilde{V}_1 \oplus d\tilde{V}_2 = U_1 \oplus U_2 = U$. Therefore e and $\delta(x)$ are represented by the same cocycle, so they are equal. Note that an obvious diagram chasing shows that $\delta(x)$ does not depend on the choice of the lifts v_1 and v_2 .

⁷In this proof, we should not use any particular realization of the complexes calculating the cohomology. Rather we will only use the *existence* of such complexes.

The trivializations can now be used to get an explicit cycle x . For this we note that the trivialization $\phi_i : F|_{B_i} \rightarrow B_i \times \mathbf{R}^n$ gives an isomorphism

$$\phi_i^* : H^*(B_i \times \mathbf{R}^n, B_i \times (\mathbf{R}^n \setminus \{0\})) \longrightarrow H^*(F|_{B_i}).$$

Note that

$$H^{*+n}(B_i \times \mathbf{R}^n, B_i \times (\mathbf{R}^n \setminus \{0\})) \cong H^*(B_i) \otimes H^n(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\}).$$

Under the isomorphism ϕ_i^* , the class u_i is the image of $1 \otimes z$, z being the generator in $H^n(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\})$. The connecting homomorphism $\delta : H^{n-1}(\mathbf{R}^n \setminus \{0\}) \rightarrow H^n(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\})$ is an isomorphism that sends t to z . The class v_i lifting u_i can be chosen to be the image of $1 \otimes t$ by ϕ_i^* . Hence v_i is the image of t by $\phi_i^* \circ p^* = (p \circ \phi_i)^*$ where $p : B_i \times (\mathbf{R}^n \setminus \{0\}) \rightarrow (\mathbf{R}^n \setminus \{0\})$ is the second projection. By definition of h , for any f in $F^0|_C$, one has the following equality $p \circ \phi_2(f) = (h \circ \pi(f)) \cdot (p \circ \phi_1(f))$ where $\pi : F^0 \rightarrow B$ is the projection. The next lemma implies that $w = v_2|_C - v_1|_C$ is equal to $(k \circ \pi)^*(t)$. Since the map $H^{n-1}(F|_C) \cong H^{n-1}(C) \rightarrow H^{n-1}(F^0|_C)$ is precisely π^* , one can set $x = k^*(t)$ for the lift of w . \square

Lemma B.8. *Let $h : D \rightarrow \mathrm{GL}^+(n, \mathbf{R})$ and $\phi : D \rightarrow \mathbf{R}^n \setminus \{0\}$ be two continuous maps. Denote by $k : D \rightarrow \mathbf{R}^n \setminus \{0\}$, $d \mapsto h(d) \cdot v_0$ an orbital application and by $h \cdot \phi$ the map $d \mapsto h(d) \cdot \phi(d)$.*

Then the following equality holds:

$$(h \cdot \phi)^* = k^* + \phi^* : H^{n-1}(\mathbf{R}^n \setminus \{0\}; \mathbf{Z}) \longrightarrow H^{n-1}(D; \mathbf{Z}).$$

Proof. Denote by o the map $\mathrm{GL}^+(n, \mathbf{R}) \rightarrow \mathbf{R}^n \setminus \{0\}$, $g \mapsto g \cdot v_0$ and by ev the map $\mathrm{GL}^+(n, \mathbf{R}) \times (\mathbf{R}^n \setminus \{0\}) \rightarrow \mathbf{R}^n \setminus \{0\}$, $(g, v) \mapsto g \cdot v$. Hence $k = o \circ h$ and $h \cdot \phi = ev \circ (h, \phi)$. By the Künneth formula, the $(n-1)$ -cohomology group of $\mathrm{GL}^+(n, \mathbf{R}) \times (\mathbf{R}^n \setminus \{0\})$ decomposes as

$$H^{n-1}(\mathrm{GL}^+(n, \mathbf{R}) \times (\mathbf{R}^n \setminus \{0\}); \mathbf{Z}) \cong H^{n-1}(\mathrm{GL}^+(n, \mathbf{R}); \mathbf{Z}) \oplus H^{n-1}(\mathbf{R}^n \setminus \{0\}; \mathbf{Z}).$$

and the map $H^{n-1}(\mathrm{GL}^+(n, \mathbf{R}); \mathbf{Z}) \oplus H^{n-1}((\mathbf{R}^n \setminus \{0\}); \mathbf{Z}) \rightarrow H^{n-1}(\mathrm{GL}^+(n, \mathbf{R}) \times (\mathbf{R}^n \setminus \{0\}); \mathbf{Z})$ is given by the two projections; its inverse is given by the inclusions of $\mathrm{GL}^+(n, \mathbf{R}) \times \{v_0\}$ and $\{g_0\} \times (\mathbf{R}^n \setminus \{0\})$. In this decomposition ev^* is the map (o^*, Id) and $(h, \phi)^*$ is $h^* + \phi^*$. Hence $(h \cdot \phi)^* = (h, \phi)^* \circ ev^* = h^* \circ o^* + \phi^* = k^* + \phi^*$. \square

REFERENCES

- [1] Dmitri V. Anosov, *Geodesic flow on closed Riemannian manifolds with negative curvature*, Proceedings of the Steklov Institute of Mathematics, No. 90 (1967). Translated from the Russian by S. Feder, American Mathematical Society, Providence, RI, 1969. MR MR0242194 (39 #3527)
- [2] Michael F. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 47–62. MR MR0286136 (44 #3350)
- [3] Michael F. Atiyah and Raoul Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615. MR MR702806 (85k:14006)
- [4] Thierry Barbot, *Quasi-Fuchsian AdS representations are Anosov*, arXiv/0710.0969, 2007.
- [5] ———, *Three-dimensional Anosov flag manifolds*, Geom. Topol. **14** (2010), no. 1, 153–191.
- [6] Nicolas Bergeron and Tsachik Gelander, *A note on local rigidity*, Geom. Dedicata **107** (2004), 111–131. MR MR2110758 (2005k:22015)
- [7] Steven B. Bradlow, Oscar García-Prada, and Peter B. Gothen, *Surface group representations and $U(p, q)$ -Higgs bundles*, J. Differential Geom. **64** (2003), no. 1, 111–170. MR MR2015045 (2004k:53142)

- [8] ———, *Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces*, *Geom. Dedicata* **122** (2006), 185–213. MR MR2295550 (2008e:14013)
- [9] ———, *Deformations of maximal representations in $\mathrm{Sp}(4, \mathbb{R})$* , arXiv/0903.5496, 2009.
- [10] Marc Burger, Alessandra Iozzi, François Labourie, and Anna Wienhard, *Maximal representations of surface groups: Symplectic Anosov structures*, *Pure Appl. Math. Q.* **1** (2005), no. 3, Special Issue: In memory of Armand Borel. Part 2, 543–590. MR MR2201327 (2007d:53064)
- [11] Marc Burger, Alessandra Iozzi, and Anna Wienhard, *Surface group representations with maximal Toledo invariant*, *C. R. Math. Acad. Sci. Paris* **336** (2003), no. 5, 387–390. MR MR1979350 (2004e:53076)
- [12] ———, *Surface group representations with maximal Toledo invariant*, arXiv/0605656, to appear in *Annals of Mathematics*, 2006.
- [13] ———, *Tight homomorphisms and Hermitian symmetric spaces*, *Geom. Funct. Anal.* **19** (2009), no. 3, 678–721. MR MR2563767
- [14] ———, *Maximal representations and Anosov structures*, in preparation, 2010.
- [15] Benson Farb and Dan Margalit, *A primer on mapping class groups*, book in preparation, v. 3.0, 2009.
- [16] Vladimir Fock and Alexander Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, *Publ. Math. Inst. Hautes Études Sci.* (2006), no. 103, 1–211. MR MR2233852
- [17] John Franks and Bob Williams, *Anomalous Anosov flows*, *Global theory of dynamical systems* (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979), *Lecture Notes in Math.*, vol. 819, Springer, Berlin, 1980, pp. 158–174. MR MR591182 (82e:58078)
- [18] Oscar García-Prada, Peter B. Gothen, and Ignasi Mundet i Riera, *Higgs bundles and surface group representations in the real symplectic group*, 2008, arXiv/0809.0576.
- [19] Oscar García-Prada and Ignasi Mundet i Riera, *Representations of the fundamental group of a closed oriented surface in $\mathrm{Sp}(4, \mathbb{R})$* , *Topology* **43** (2004), no. 4, 831–855. MR MR2061209 (2005k:14019)
- [20] William M. Goldman, *The symplectic nature of fundamental groups of surfaces*, *Adv. in Math.* **54** (1984), no. 2, 200–225. MR MR762512 (86i:32042)
- [21] ———, *Geometric structures on manifolds and varieties of representations*, *Geometry of group representations* (Boulder, CO, 1987), *Contemp. Math.*, vol. 74, Amer. Math. Soc., Providence, RI, 1988, pp. 169–198. MR MR957518 (90i:57024)
- [22] ———, *Topological components of spaces of representations*, *Invent. Math.* **93** (1988), no. 3, 557–607. MR MR952283 (89m:57001)
- [23] William M. Goldman, Michael Kapovich, and Bernhard Leeb, *Complex hyperbolic manifolds homotopy equivalent to a Riemann surface*, *Comm. Anal. Geom.* **9** (2001), no. 1, 61–95.
- [24] Peter B. Gothen, *Components of spaces of representations and stable triples*, *Topology* **40** (2001), no. 4, 823–850. MR MR1851565 (2002k:14017)
- [25] Olivier Guichard, *Composantes de Hitchin et représentations hyperconvexes de groupes de surface*, *J. Differential Geom.* **80** (2008), no. 3, 391–431. MR MR2472478 (2009h:57031)
- [26] Olivier Guichard and Anna Wienhard, *Convex foliated projective structures and the Hitchin component for $\mathrm{PSL}_4(\mathbb{R})$* , *Duke Math. J.* **144** (2008), no. 3, 381–445. MR MR2444302
- [27] ———, *Domains of discontinuity with compact quotient and Anosov representations*, in preparation, 2010.
- [28] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR MR1867354 (2002k:55001)
- [29] Luis Hernández, *Maximal representations of surface groups in bounded symmetric domains*, *Trans. Amer. Math. Soc.* **324** (1991), 405–420. MR MR1033234 (91f:32040)
- [30] Nigel J. Hitchin, *The self-duality equations on a Riemann surface*, *Proc. London Math. Soc.* (3) **55** (1987), no. 1, 59–126. MR MR887284 (89a:32021)
- [31] ———, *Lie groups and Teichmüller space*, *Topology* **31** (1992), no. 3, 449–473. MR MR1174252 (93e:32023)
- [32] Dale Husemoller, *Fibre bundles*, second ed., Springer-Verlag, New York, 1975, *Graduate Texts in Mathematics*, No. 20. MR MR0370578 (51 #6805)
- [33] Nikolai V. Ivanov, *Mapping class groups*, *Handbook of geometric topology*, North-Holland, Amsterdam, 2002, pp. 523–633. MR MR1886678 (2003h:57022)
- [34] Dennis Johnson, *Spin structures and quadratic forms on surfaces*, *J. London Math. Soc.* (2) **22** (1980), no. 2, 365–373. MR MR588283 (81m:57015)

- [35] Anthony W. Knap, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston Inc., Boston, MA, 2002. MR MR1920389 (2003c:22001)
- [36] François Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), no. 1, 51–114. MR 2221137
- [37] ———, *Cross ratios, Anosov representations and the energy functional on Teichmüller space*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), no. 3, 437–469. MR MR2482204
- [38] Jun Li, *The space of surface group representations*, Manuscripta Math. **78** (1993), no. 3, 223–243. MR MR1206154 (94c:58022)
- [39] Quentin Mérigot, *Anosov AdS representations are quasi-Fuchsian*, arXiv/0710.0618, 2007.
- [40] John Milnor, *On the existence of a connection with curvature zero*, Comment. Math. Helv. **32** (1958), 215–223. MR MR0095518 (20 #2020)
- [41] André Gama Oliveira, *Representations of surface groups in the projective general linear group*, 2009, arXiv/0901.2314.
- [42] Norman Steenrod, *The Topology of Fibre Bundles*, Princeton Mathematical Series, vol. 14, Princeton University Press, Princeton, N. J., 1951. MR MR0039258 (12,522b)
- [43] Emery Thomas, *On tensor products of n -plane bundles*, Arch. Math. **10** (1959), 174–179. MR MR0107234 (21 #5959)
- [44] Domingo Toledo, *Representations of surface groups in complex hyperbolic space*, J. Differential Geom. **29** (1989), no. 1, 125–133. MR MR978081 (90a:57016)
- [45] Anna Wienhard, *The action of the mapping class group on maximal representations*, Geom. Dedicata **120** (2006), 179–191. MR MR2252900 (2008g:20112)

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